INTERNATIONAL SCHOOL
FOR
ADVANCED STUDIES
(SISSA-ISAS) - TRIESTE

Thesis submitted for the degree of Doctor
Philosophiæ

Discrete Approaches Towards the Definition of a Quantum Theory of Gravity

Candidate
Gabriele Gionti

Supervisors
Prof. Mauro Carfora
Prof. Alessandro D'Adda

Academic year 1997/98

Acknowledgments

I am grateful to Prof. Mauro Carfora for introducing me to the subject of Dynamical Triangulations, for collaboration and discussions. I am indebted also with Prof. Alessandro D'Adda for collaboration on Regge calculus and friendship toward me. Many thanks to Dr. Ruth M. Williams for having read this thesis and for remarks and questions which have improved it a lot. It is a pleasure to acknowledgment Davide Gabrielli for daily discussions and collaborations. I have shared with him my experience in Sissa. I want to single out all my friends, now spread in the world, with whom I began my experience in Sissa: Carlo Acerbi, Francesca Colaiori, Paolo De Los Rios, Alessandro Flammini, Beppe Turco. In particular Stefano Panzeri and Loredana Fantauzzi and their baby (not "universe") Matteo (also if up to now I have not seen him), my friendship with them is one of the best thing in these years. I cannot forget Azer Akhmedov (Azeglio) and his moustaches which gave him the appearance of an Islamic fundamentalist. It is still surprising for me how he is able to understand mathematical concepts and misunderstand every other thing. A special mention is for Antonio Liguori, outstanding scientist, with whom I have had many scientific discussions, and whose friendship I enjoyed this last year in Sissa. He has been the only person who has given attention (and I think that this is something which could be used against him) to my tentative to interpret our world in a Platonic sense. I think that in order to verify these interpretations one time he conducted me in the most dangerous streets of Naples.

I want to remember Fabrizio Melchionna and his nostalgia of Ragusa. The two astronomers Michele Maris and Cesario Lia and our metaphysical discussions in which each one of us remains always on his own position. Last but not least are Marco Bertola, Stefano Bianchini, and Davide Guzzetti with whom, recently, I have daily scientific and not "strictly" scientific discussions.

This thesis and my scientific work in Sissa is dedicated to my mother Angela, my brother Oreste and my uncle Attilio whose material life has been recently destroyed by the cancer

Contents

Chapter 1

Introduction

In spite of many recent developments, in particular in string theory, the problem of quantization of General Relativity is still an open one both from mathematical and physical point of view.

In this thesis we discuss some contributions to the lattice approach to Euclidean Quantum Gravity, namely to Regge Calculus and Dynamical Triangulations that offer the most natural discretization of General Relativity. This discretization consists in going from a Riemannian manifolds to triangulations of Piecewise-Linear manifolds. In both cases the partition function is a sum over the triangulations of Piecewise-Linear manifolds. Each triangulation is weighted by a factor equal to the exponential of minus the discretized version of the Einstein-Hilbert action (Regge-Einstein action). Moreover the diff-invariant continuum measure on the Riemannian structures of a manifold M is replaced, in general, by a DeWitt-like measure for the edge lengths in Regge Calculus and by a micro-canonical measure in the dynamical triangulations.

The bulk of this thesis is based on two original contributions which are reported in chapter three and four, while in chapter two we introduce very briefly the main notions about the model of dynamical triangulations. All the notions that we give in this chapter are well known and established in the literature and we introduce them since they will be used in chapter three.

Along the main stream of the connection between Euclidean quantum field theory and classical statistical mechanics we introduce the notions of micro-canonical, canonical, and grand-canonical partition functions. In particular in two dimensions we mention different analytical ways in which the number of combinatorial inequivalent triangulations was calculated. Recent results on the estimate of micro-canonical partition function in three and four dimensions are summarized.

This estimate will be extensively used for the classification of the elongated phase of dynamical triangulations in four dimensions in chapter three.

Successively we give the definition of the two point Green function in the context of dynamical triangulations. As a consequence of the definition of the Green function, we give the definition of the susceptibility function. From its behavior near the critical line we define the string susceptibility exponent γ_{str}.

In chapter three we shall study the elongated phase of dynamical triangulations in four dimensions. We begin by introducing Walkup's theorem which characterize the triangulations with the topology of the sphere in four dimensions. We review the kinematical bounds which are fixed by this theorem. Furthermore by using the expression of the estimated canonical partition function, we stress that the average curvature is saturated in correspondence to the kinematical bound of Walkup's theorem. This means that for values of k_2, the inverse of gravitational constant, greater than the value k_2^c for which the Walkup bound is reached, there is in the statistical ensemble of equilateral triangulations of the sphere S^4 a prevalence of particular triangulations called "Staked Spheres". These configurations are the only triangulations for which the Walkup bound is realized. They have a simple tree-like structure that can be mapped into branched polymers structures. Anyway the map is not one to one in the sense that combinatorial inequivalent stacked spheres can be mapped into the same branched polymer. We recognize that a stacked sphere fits with the model of a network of baby universes which has been formulated from the analysis of the results of the Monte Carlo simulations in four dimensions.

We construct two distinct models of branched polymers and we put them in correspondence with the stacked spheres by the dual map. This analysis shows that the string susceptibility of the stacked spheres is less than 1.

At the end we analyse a model taken from the theory of random surfaces and adapted to the stacked spheres. The aim of this analysis is to show some evidences on the analogy between the stacked spheres and self-energy Feynman graphs relative to matrix models of two dimensional triangulations. What we learn from this analysis and from all previous considerations is that there is a strong evidence that the stacked spheres correspond to a mean field phase in which the string susceptibility exponents is $\gamma_s = \frac{1}{2}$, so that any attempt of performing a continuum limit in this phase will give, even if we assume the convergence of the Schwinger functions, a Gaussian measure.

Recent numerical evidence of a first order phase transition of the model of dynamical triangulations in four dimensions and the previous strong evidence of

a trivial elongated phase suggest that more efforts towards new discrete models for simplicial quantum gravity might be required.

Along this line and following the work of various authors we study Regge calculus as a local theory of Euclidean Poincaré group in the first order formalism. The reason for a first order formalism is both theoretical and technical. With respect to traditional Regge Calculus the novelty lies both in its formulation as a gauge theory and in the first order formalism. The gauge theory approach results mainly in the deficit angle being replaced by its sine. The first order formalism has others effect of smoothing out some pathological configurations, like "spikes", which might prevent the theory from having a smooth continuum limit. These configurations are in fact in the region of large deficit angles where the first order formalism and the second order formalism are not equivalent on a lattice.

We first review and improve some definitions of a previous work on this subject. In particular it is stressed how a group theoretical formulation of Regge calculus allows to write an action on the dual Voronoï complex of the original simplicial complex which is quite similar to a gauge theory and, more precisely,

looks like the Wilson action for lattice gauge theory.

We prove that this action does not depend from the orientation of the Voronoï plaquette.

We formulate a first order principle in which we have two sets of independent variables: the normals to the $n-1$-faces and the connection matrices. The normals are considered as the analogous of the n-bein in the continuum theory, and the connection matrices as the connection one form in General Relativity. The main result of this chapter is that we prove in the case of "small deficit angles" that Regge calculus is a solution of the first order formalism. This result is not obvious if we vary indipendently the two sets of variables above.

Then we derive the general field equations for the connection matrices and for the normals. We use the method of Lagrange multipliers to take in account the constraints of the theory. We propose a method for the calculation of the Lagrange multipliers by using the one to one correspondence among the normals to the faces of the n simplices and the circumcentric coordinates of the vertices.

A measure for the path integral for this simplicial theory of gravity is introduced and it is shown that it is locally invariant under $SO(n)$. As a last step we propose a coupling of this lattice theory of gravity with fermionic matter. This coupling is entirely performed by following the general prescription of the continuum theory. In other formulation of discrete gravity (Regge calculus and dynamical triangulations) the coupling with fermionic matter is usually introduced "ad hoc". In

this approach the coupling with fermionic matter is given by considering spinorial representation of $SO(n)$.

Chapter 2

Dynamical Triangulations

2.1 Introduction

In this chapter we shall introduce the basic tools for the simplicial approach [12] [13] [11] to Euclidean Quantum Gravity (see [2] for the main articles on the subject and also reference [3]) *via* the theory of Dynamical Triangulations [30] [64] [67] [66]. A precursor of Dynamical Triangulations has been Weingarten [25]. Römer and Zähringer [29] proposed for the first time this model as a gauge fixing of Regge calculus. In section one we begin by considering the class of equilateral triangulations of Picewise-Linear (PL) manifolds (for a review on PL-manifods see [7] [5] [6]) which are used in dynamical triangulations. Successively we define the action for dynamical triangulations as a restriction of the Regge-Einstein action [1] to the equilateral triangulations of PL-manifolds. The partition function for dynamical triangulations is defined over the ensemble of equilateral triangulations of PL-manifolds.

In this framework we address the counting problem of the number of combinatorial inequivalent triangulations and we briefly review the different methods of enumeration in the two dimensional case. In three and four dimensions some recent analytical results are illustrated [30] [31] [32] [33].

In section 2.21 following [64] we give the definition of the Green functions in the framework of dynamical triangulations. The exposition always follows the connection with classical statistical mechanics and at the end we give the definition of susceptibility function as a direct consequence of the definition of the grand-canonical Green function.

All this chapter has to be considered as an introduction to the main concepts of dynamical triangulations which will be used in the study of the elongated phase

in 4-D dynamical triangulations.
Anyway it is important in our opinion to remark that the whole approach of dynamical triangulations must be considered in the general framework of Euclidean Lattice field theory as explained in [36] chap. 15. It is crucial in this approach to determine the Green functions (the Schwinger functions) and to look for a second order phase transition in the parameter space. If we have a second order phase transition then the quantum theory of gravity may be defined at the critical point by looking for the scaling limit of the Green functions. If this limit exists we ask if it fits with the Osterwalder-Schrader axioms [34] [35]. In the affermative case the Riemannian Green functions are the Wick rotated version of the Lorentian Green Function (the Wightman functions) [37] [38]. A first implementation of the above ideas of Euclidean lattice field theory to simplicial quantum gravity has been given by Rocek and Williams [12] [13].

It can happen (as in the case of $\lambda\phi^4$) that the continuum limit gives a gaussian measure (free theory), then the theory is trivial. We want to stress that the final goal of these theories is to find a non trivial continnum limit.

2.2 The Model of the Dynamical Triangulations

The standard rule in Dynamical Triangulations (for a review on the recent results see [21] and also [80]) is to consider all the triangulations of PL-manifolds made by equilateral simplices of fixed edge lengths, say a. This implies that the geometrical structures are even more rigid with respect to Regge Calculus [1] (for a recent review on Regge calculus see [23]). The set of piecewise-linear maps on these simplicial complexes depend only by their combinatorial structures. So that two triangulations T_a and T'_a are equivalent if there is a piecewise linear map ϕ between them such that it maps one to one the vertices of T_a into the vertices of T'_a in such a way that $(\phi(v_i), \phi(v_j))$ is an edge of T'_a if and only if (v_i, v_j) is an edge of T_a and so on for every simplex of any dimension.
The $n-2$ simplices are called *bones* B. The dihedral angle [1] of a n simplex σ^n on a bone is $\cos^{-1}\frac{1}{n}$. If we indicate by $q(B)$ the number of n-dimensional simplices which share the bone B, the Regge curvature [1] on the bone B is

$$K(B) = 2\left(2\pi - q(B)\cos^{-1}\frac{1}{n}\right)V(B) \tag{2.1}$$

where $V(B)$ is the $n-2$-dimensional volume of the bone. By standard formula we know that the volume of a n-dimensional equilateral simplex $V(\sigma^n)$ is

$$V(\sigma^n) = \frac{a^n\sqrt{n+1}}{(n)!\sqrt{2^n}} \quad . \tag{2.2}$$

The Regge-Einstein action with cosmological term for dynamical triangulations T_a without boundary can be seen as a functional of the form

$$S_R(\Lambda, G, T_a) \equiv \frac{\Lambda}{8\pi G}\sum_{\sigma^n} V(\sigma^n) - \frac{1}{16\pi G}\sum_B K(B) \quad , \tag{2.3}$$

where the first sum is over all n-dimensional simplices of the triangulation T_a.

We define this two bare coupling constants

$$\begin{aligned} k_n &\equiv \frac{\Lambda}{8\pi G}V(\sigma^n) + n(n+1)\frac{\cos^{-1}\frac{1}{n}}{16\pi G}V(B) \\ k_{n-2} &\equiv \frac{V(B)}{4G} \quad , \end{aligned} \tag{2.4}$$

then the action on a triangulation T_a takes the standard form

$$S(k_n, k_{n-2}, T_a) = k_n N_n - k_{n-2}N_{n-2} \tag{2.5}$$

where N_n and N_{n-2} are the numbers, respectively, of the n and $n-2$ simplices of the triangulation T_a.

The partition function on the triangulation T_a of a PL-manifolds $\mathcal{P}L$ is defined as

$$Z(\mathcal{P}L, k_n, k_{n-2}) \equiv \sum_{T_a} e^{-k_n N_n + k_{n-2}N_{n-2}} \quad . \tag{2.6}$$

Notice that all the triangulations of the PL-manifold $\mathcal{P}L$, have the same weight in the path-integral (for a discussion about the measure of dynamical triangulations see [22]) .

The partition function 2.6 can be rewritten in the following way

$$Z(k_n, k_{n-2}) \equiv \sum_{\mathcal{P}L}\sum_{N_n} e^{-k_n N_n} \sum_{N_{n-2}} e^{k_{n-2}N_{n-2}} \sum_{\#T_a(\mathcal{P}L, N_n, N_{n-2})} 1 \tag{2.7}$$

where $\sum_{\mathcal{P}L}$ means the sum over all piecewise-flat topologies and $\sum_{\#T_a(\mathcal{P}L,N_n,N_{n-2})}$ is the number of combinatorial inequivalent equilateral triangulations $T_a(\mathcal{P}L, N_n, N_{n-2})$ of a fixed PL-topology, fixed number N_n of n-simplices and fixed number N_{n-2} of $n-2$-simplices. Now we can consider

$$\rho_a(\mathcal{P}L, N_n, N_{n-2}) \equiv \sum_{\#T_a(\mathcal{P}L, N_n, N_{n-2})} 1 \tag{2.8}$$

as the micro-canonical partition function over the ensemble of the equilateral triangulations of edge length a with fixed PL-topology. The following formula

$$Z(\mathcal{P}L, N_{n-2}, k_{n-2}) \equiv \sum_{N_{n-2}} e^{k_{n-2}N_{n-2}} \rho_a(\mathcal{P}L, N_n, N_{n-2}) \tag{2.9}$$

is the canonical partition function, in which $\sum_{N_{n-2}} e^{k_{n-2}N_{n-2}}$ plays the role of the Gibbs measure. Finally

$$Z(k_n, k_{n-2}) \equiv \sum_{\mathcal{P}L} \sum_{N_n} e^{-k_n N_n} Z(N_n, k_{n-2}) \tag{2.10}$$

is the grand-canonical partition function in which $e^{-k_n N_n}$ can be considered as the equivalent of the chemical potential. Anyway since even in two dimensions the sum over the PL-topologies is divergent (see ref. [43] for a brief account), many times we will restrict to the only topology of the sphere in every dimension.

Let's start to examine the two dimensional case. The Dehn-Sommerville equations (see [65] p. 62 and also [60] p. 80) are:

$$\begin{aligned} N_2 - N_1 + N_0 &= \chi\left(T_a(\mathcal{P}L)\right) \\ 2N_1 &= 3N_2 \quad , \end{aligned} \tag{2.11}$$

where $\chi\left(T_a(\mathcal{P}L)\right)$ is the Euler-Poincaré characteristic of the PL-manifold $\mathcal{P}L$ whose triangulation is $T_a(\mathcal{P}L)$. As is well known in two dimensions, topological manifolds and PL-manifolds are equivalent and the topology, for compact, connected and orientable two-dimensional manifolds , is completely classified by the genus g of the manifold [39]. From the Dhen-Sommerville equations 2.12 we can express all the components of the f-vector (see [60] p. 78) as function of N_2 and $\chi\left(T_a(\mathcal{P}L)\right)$, that is to say

$$\begin{aligned} N_0 &= \frac{N_2}{2} + \chi\left(T_a(\mathcal{P}L)\right) \\ N_1 &= \frac{3}{2} N_2 \quad . \end{aligned} \tag{2.12}$$

The asymptotic number of combinatorial inequivalent triangulations of the sphere S^2 was calculated for the first time by the mathematician Tutte [40]. The

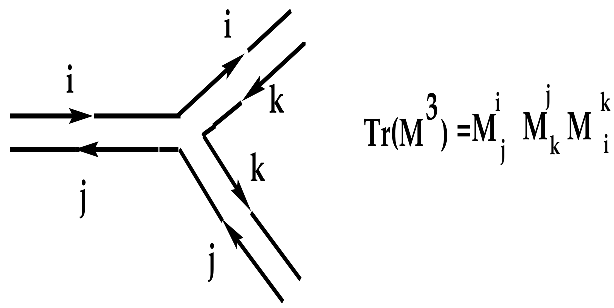

Figure 2.1: Feynman graph for the matrix model relative to the triangulations. Each line has two entries one for the row and other for the column of the matrix M. In each vertex there are always three double lines

underlying idea has been to map by the stereographic projection the triangulations of the sphere in planar triangulations and to enumerate them by the techniques of generating series built up by using the geometrical properties of the planar triangulations. The result is the following

$$\rho_a(S^2, N_2) \asymp N_2^{-\frac{7}{2}} e^{k_2^c N_2} \quad , \tag{2.13}$$

where k_2^c is a numerical constant. It is important that the growth of the number of the triangulations is at most exponential. In fact in the opposite case the divergences make it impossible to define a statistical theory like 2.7. The result 2.13 can be obtained again by using the quantum field theory techniques of the matrix models [41]. This techniques is based on the use of the generating functional

$$Z_N(g) = \int dM \, exp \left(-\frac{1}{2} tr(M^2) - \frac{\lambda}{\sqrt{N}} tr(M^3) \right) \tag{2.14}$$

in which M is an $N \times N$ Hermitian matrix and λ the coupling constant, dM is the Haar-measure on this matrix group. The path-integral 2.14 will generate a perturbative series whose Feynman diagrams can be represented as double line one for each index of the matrix M_{ij} (see figure 2.1).

Since in the equation 2.14 there is a trace of the product of matrices, the Feynman diagrams have to be closed graphs. The fact that there are two lines and the possibility of all combinations of the indices means that in general the graphs can be closed only on Riemann surfaces (see figure 2.2). Furthermore each Feynman graph corresponds to a dual triangulation.
It can be show that the perturbative series in λ relative to connected Feynman

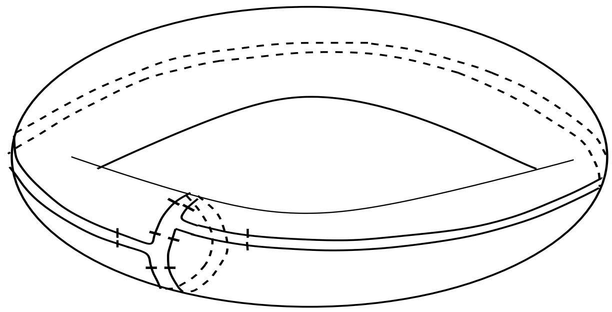

Figure 2.2: An example of a second order perturbation term which is rappresented by two double graphs of $\mathrm{Tr}(M^3)$ which can be closed on a two dimensional torus

double graphs generated by 2.14 can be arranged as power serie in N of the genus g of the Riemann surfaces. It follows that the coefficients of this series give the generating functions for the number of combinatorial inequivalent dual triangulations of a fixed topology. Since there is a one to one correspondence between any triangulation and its (topological or metrical) dual we can obtain the generating functions of the combinatorial inequivalent triangulations for any topology (see [41] and also [42]).

Another derivation of the formula 2.13 has been recently found [30] and is based on the enumeration of the curvature assignment of the bones of the triangulation. This estimate can be

extended to all two-dimensional topologies and to higher dimensions.

It follows that the two dimensional grandcanonical partition function can be written as

$$Z(k_2, k_0) = \sum_g e^{k_0 \chi(g)} \sum_{N_2} N_2^{\gamma_{str}-3} e^{-(k_2 - \frac{k_0}{2} - k_2^c)N_2} \tag{2.15}$$

where $\chi(g) = 2 - 2g$. $k_2 - \frac{k_0}{2} - k_2^c$ is called the critical line since for $k_2 > \frac{k_0}{2} + k_2^c$ the partition function is convergent and for $k_2 < \frac{k_0}{2} + k_2^c$ the partition function is divergent. γ_{str} is the exponent of the string susceptibility. This exponent, that in two dimensions is equal to $-\frac{1}{2}$, controls the divergent part of the partition function. More precisely if we fix the genus g and consider the limit $k_2 \mapsto \left(\frac{k_0}{2} + k_2^c\right)^+$, we have that from 2.15

$$\lim_{k_2 \mapsto \left(\frac{k_0}{2} + k_2^c\right)^+} Z_g(k_2, k_0) = Z_g^{reg} + (k_2 - \frac{k_0}{2} - k_2^c)^{2-\gamma_{str}} \tag{2.16}$$

with Z_g^{reg} the finite part (if any) of the partition function near the critical line. It is now clear that the discontinuity points of the partition function are on this line, so an eventual critical point is on this line.
As regard the sum over the genus g of the surface it is divergent and even not Borel summable. There is a (non-rigorous) way to define a renormalizzation for this series known as double scaling limit (see [43] for a brief account). Anyway, in general, we will restrict to a given topology, for example the sphere S^2.

In three and four dimensions in the case of the sphere topology the estimate [30] gives, for the microcanonical partition function, an exponential bound. In particular by posing $N_{n-2} = N_n \eta$ we have that [30]

$$\rho_a(S^n, N_n, N_{n-2}) = g(N_n, \eta) e^{N_n q(\eta)}, \quad n = 3, 4 \quad , \tag{2.17}$$

where $g(N_n, \eta)$ has a subleading growing in N_n respect to the exponential factor $e^{N_n q(\eta)}$. As we will see in the following geometrical arguments fix the range of η in the closed interval $[\eta_1, \eta_2]$. So that using the Euler-Maclaurin summation formula [44] we can approximate the sum by the integral so that the canonical partition function is

$$Z(S^n, N_n, k_{n-2}) \asymp \int_{\eta_1}^{\eta_2} g(N_n, \eta) e^{N_n q_1(\eta, k_{n-2})} d\eta \tag{2.18}$$

where $q_1(\eta, k_{n-2}) = q(\eta) + \eta k_{n-2}$. We can use the Laplace method for giving an asymptotic estimate of the integral 2.18. We have to compute the point of absolute maximum η^* of the function $q_1(\eta, k_{n-2})$ in the interval of integration. In general this point will be a function of k_{n-2}, that is $\eta^* = \eta^*(k_{n-2})$. So we have

$$Z(S^n, N^n, k_{n-2}) \asymp g(N_n, \eta^*(k_{n-2})) e^{N_n q_1(\eta^*(k_{n-2}), k_2)} \quad . \tag{2.19}$$

We define $k_n^c(k_{n-2}) \equiv q_1(\eta^*(k_{n-2}), k_2)$, so that the grand-canonical partition function is

$$Z(S^n, k_n, k_{n-2}) \asymp \sum_{N_n} g(N_n, \eta^*(k_{n-2})) e^{-N_n(k_n - k_n^c(k_{n-2}))} \quad . \tag{2.20}$$

It is clear that now the critical line is $k_n = k_n^c(k_{n-2})$ with the same meaning as in the two dimensional case.

2.3 Green Functions in Dynamical Triangulations

Since the geometry in General Relativity has a dynamical role, the definition of the Green function in Quantum Gravity is different from ordinary quantum field theory.
The formal continuum definition of the unnormalized two point function is the following

$$G_2(\Lambda, G, r) \equiv \sum_{\mathrm{Top}(M)} \int \mathcal{D}(g) e^{-S_{E-H}(M,g)}$$
$$\int_{(M,g)} d^n y \sqrt{\det g(y)} \int_{(M,g)} d^n x \sqrt{\det g(x)} \delta(d_g(x,y) - r) \qquad (2.21)$$

where Λ and G are respectively the cosmological and gravitational constant, the sum is over the topological structures of the differentiable manifold M, the measure $\mathcal{D}(g)$ is over the Riemannian structure allowed by M, S_{E-H} is the Einstein-Hilbert action and $d_g(x,y)$ is the geodesic distance between the points x and y on the Riemannian manifold (M,g). To normalize $G_2(\Lambda, G, r)$ we have to divide it by the partition function. So in this definition we have to perform a sort of average over all points of the Riemannian manifold (M,g) that are at distance r. In the discrete we cannot transfer *verbatim* the above definition due to the coordinate invariance of the dynamical triangulations. It is necessary to define the two point function in such a way that the analogous quantities, in the discrete, of x and y in the continuum have a dependence from the cut-off a.

A path in a triangulation T_a of a n-dimensional PL-manifolds is a sequence $\{S_j\}_{j=1}^{l}$ of l n-dimensional simplices with the property that S_j and S_{j+1} ($S_{j+1} \neq S_j$) have a common face, that is to say $I(S_j, S_{j+1}) = 1$ (see [8] p. 526). Suppose that the path does not intersect itself, that is to say $I(S_p, S_q) = 1$ if and only if or $q = p - 1$ or $q = p + 1$, we call it a simple path. In this case we can define the length of the path $\{S_j\}_{j=1}^{l}$ as the number of the faces that the l simplices have in common, that are $l - 1$. This definition comes from graph theory, in fact if we think to the dual of the path $\{S_j\}_{j=1}^{l}$, every simplex is mapped into a vertex, and the face between two simplices is mapped into a edge joining the two vertices. So the length of the path is equal to the number of the dual edge of the path (see fig. 2.3).
We define the distance between two simplices A and B of the triangulation T_a as the length of the simple path which has the minimal length among all simple paths $\{S_j\}_{j=1}^{l_q}$ such that $S_1 = A$ and $S_{l_q} = B$.

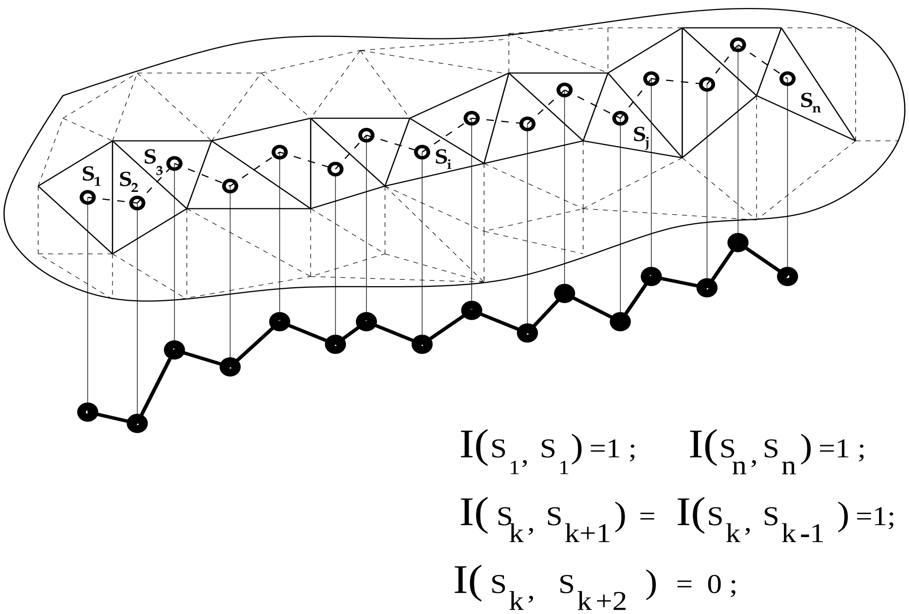

Figure 2.3: A simplicial path between the two simplices S_1 and S_n of a triangulation. The simplicial path is simple and its dual is obtained by joining the baricenters or the circumcenters of the path's simplices. On this path the properties of the incidence matrix I are illustrated

Let's now give the definition of unnormalized micro-canonical Green function. The unnormalized micro-canonical Green function is defined as [64]

$$G(\mathcal{P}L, r, N_n, N_{n-2}) \equiv \sum_{\#T_a(\mathcal{P}L,r,N_n,N_{n-2})} 1 \tag{2.22}$$

where $\sum_{\#T_a(\mathcal{P}L,r,N_n,N_{n-2})}$ is the number of inequivalent equilateral triangulations $T_a(\mathcal{P}L, r, N_n, N_{n-2})$ of a PL-manifold $\mathcal{P}L$ with two labelled n-simplices at fixed distance r and with fixed number N_n and N_{n-2} of respectively n and $n-2$ simplices. The normalized partition function is obtained by dividing 2.22 by the microcanonical partition function. So on we have that the canonical Green function is

$$G(\mathcal{P}L, r, N_n, k_{n-2}) \equiv \sum_{N_{n-2}} e^{-k_{n-2}N_{n-2}} G(\mathcal{P}L, r, N_n, N_{n-2}) \tag{2.23}$$

and finally the grand-canonical Green function

$$G(\mathcal{P}L, r, k_n, k_{n-2}) \equiv \sum_{N_n} e^{k_n N_n} G(\mathcal{P}L, r, N_n, k_{n-2}) \quad . \tag{2.24}$$

From the grand-canonical Green Function we can define another quantity often used in the literature the *susceptibility* $\chi(\mathcal{P}L, k_n, k_{n-2})$ which is defined in the following way

$$\chi(\mathcal{P}L, k_n, k_{n-2}) \equiv \sum_{r=0}^{\infty} G(\mathcal{P}L, r, k_n, k_{n-2}) \quad . \tag{2.25}$$

In the previous definition we can exchange the sum over r with all the sums in the definition of the grand-canonical Green function 2.24 up the micro-canonical Green function 2.22. So that we will have

$$\chi(\mathcal{P}L, k_n, k_{n-2}) = \sum_{N_n} e^{k_n N_n} \sum_{N_{n-2}} e^{-k_{n-2}N_{n-2}} \sum_{r} \sum_{\#T_a(\mathcal{P}L,r,N_n,N_{n-2})} 1 \quad . \tag{2.26}$$

The last two sums in the left hand side of the previous equation are the number of inequivalent triangulations with two labelled n-dimensional simplices. For large value of N_n we will expect that asymmetrical triangulations will prevail in the micro-canonical ensemble (this is, for example, true for two-dimensional convex polyhedra [45], for same particular planar triangulations [46] and planar maps [47] and it is conjectured for all planar maps. In four dimensions this is verified, so to

say, indirectly because once we do this *antsaz* and derive the formula for the number of *baby universes*(see appendix B) it is in good agreement with the numerical simulations [48] [52] [55]). In this hypothesis for each labelling of a n-simplex we will produce a number of combinatorially inequivalent triangulation that asymptotically is $N_n \rho_a(\mathcal{P}L, N_n, N_{n-2})$. Finally the leading asymptotic estimate of the number of combinatorially inequivalent triangulations with two labelled simplices is

$$\sum_{r} \sum_{\#T_a(\mathcal{P}L, r, N_n, N_{n-2})} \asymp N_n^2 \rho_a(\mathcal{P}L, N_n, N_{n-2}) \tag{2.27}$$

Then the susceptibility 2.25 is equal to

$$\chi(\mathcal{P}L, k_n, k_{n-2}) \asymp \sum_{N_{n-2}} N_n^2 e^{-k_n N_n} Z(\mathcal{P}L, N_n, k_{n-2}) \tag{2.28}$$

so that

$$\chi(\mathcal{P}L, k_n, k_{n-2}) \asymp \frac{\partial^2}{\partial k_{n-2}^2} Z(k_n, k_{n-2}, \mathcal{P}L) \quad . \tag{2.29}$$

Near the critical line $k_n \mapsto (k_n^c(k_{n-2}))^+$ we will have the following asymptotic behaviour of the susceptibility

$$\lim_{k_n \to (k_n^c(k_{n-2}))^+} \chi(\mathcal{P}L, k_n, k_{n-2}) \asymp \chi^{reg} + \frac{1}{(k_n - k_n^c(k_{n-2}))^{\gamma_{str}}} \tag{2.30}$$

where χ^{reg} is the finite part (if any) of the susceptibility.

Chapter 3

The Geometry of the Elongated Phase of 4-D Simplicial Quantum Gravity

3.1 Introduction

In this chapter we are going to discuss in detail the elongated phase of 4-D dynamical triangulations. The early numerical simulations in four dimensions have produced some evidence of the presence of two distinct phases in the ensemble of four dimensional triangulations: the crumpled phase and the elongated phase [49] [50] [51] (it is interesting to remark that this picture emerges also in the simulations of 4D quantum Regge calculus [15] [16] [17]). The crumpled phase is characterized by the presence of few bones and many four simplices incident on them. There are very few baby universes (see appendix B). The elongated phase is characterized by the presence of many bones and of many baby universes of smallest size (blips). These facts have encouraged people to use the relative abundance of baby universes as an order parameter to distinguish these two phases.
Anyway it was still unclear why in the elongated phase there is an upper kinematical bound on the values of $\eta = \frac{N_2}{N_4}$ and a prevalence of simple tree-graphs (branched polymers) which appear as a model of proliferating baby universes [59]. Established in [30] in section 3.2 we explain that the upper kinematical bound on the values of η is due to the Walkup's theorem for the triangulations of the four sphere. In particular this theorem fixes the upper value of η in the large-N_4 limit $N_4 \mapsto \infty$. In section 3.3 we introduce a recent [30] [92] analytical estimate for the canonical partition functions for dynamical triangulations in four dimensions relative to the triangulations of the sphere S^4. This estimate has the right behaviour

in the limit $N_4 \mapsto \infty$, because the average curvature 3.16 in the elongated phase remains constant in perfect agreement with Walkup's theorem. Furthermore this theorem says also that the dominating configurations in this phase are "Stacked Spheres". In section 3.4 we discuss how the stacked spheres have a simple tree structure and can be mapped into branched polymers structures (see appendix A). However the problem arises that the map is not a one to one, since there are more stacked spheres than branched polymers. So in section 3.5 we consider the action of dynamical triangulations restricted to stacked spheres and show that the partition function over the ensemble of the stacked spheres is equal or greater than the partition function of branched polymers which are the immages of the above map, and equal or less than a particular model of branched polymers. This argument shows that the susceptibility exponent γ_s is less than one. To show that $\gamma_s = \frac{1}{2}$ we use an argument taken from the physical literature [64] [67]. This argument follows the line of a model introduced for the first time in the theory of random surfaces [63].
We take into account some particular kind of non standard triangulations and put it in correspondence with the model of stacked spheres considered as a model of proliferating baby universes. Physical arguments ensure that the two models belong to the same universality class and we show that the susceptibility exponent is $\gamma_s = \frac{1}{2}$ as for the branched polymers.
These types of analytical results were already well known from Monte Carlo simulations. An early analytical attempt, in the direction of interpreting the elongated phase as a stacked spheres phase, has been done in reference [58] in which a map from a sort of stacked sphere triangulations to branched polymers is considered. Anyway the deep reason for polymerization mechanism and with the kinematical bound on the values of η is given only by Walkup theorem.

All these facts point out that the elongated phase of 4-D dynamical triangulations is a mean field phase, so that we expect that if we take the continuum limit in the framework of Euclidean lattice field theory (c.f. [36] chap. 15), in this phase, we will obtain a probabilistic Feynman measure which is Gaussian, or, in other words, a trivial theory

3.2 Walkup's Theorem

As anticipated in section 2.2 of chapter 2 due to geometrical constraints the variable η in four dimensions can vary in a finite interval $[\eta_1, \eta_2]$. The limits η_1 and η_2 of this interval are fixed respectively by the following Walkup theorem and

Dehn-Sommerville equations [60].

Let us label the bones [69] by an index α and denote by $q(\alpha)$ the number of four simplexes incident on it; the average number of simplexes incident on a bone is

$$b \equiv \frac{1}{N_2}\sum_{\alpha} q(\alpha) = 10\left(\frac{N_4(T_a)}{N_2(T_a)}\right) \tag{3.1}$$

since each simplex is incident on 10 different bones. b ranges between two kinematical bounds which follow from Dehn-Sommerville and a theorem by Walkup [60]. This latter theorem is also relevant in classifying the "elongated phase" of four-dimensional simplicial quantum gravity as we shall see below

Theorem 3.2.1 *If T is a triangulation of a closed, connected four-dimensional manifold then*

$$N_1(T) \geq 5N_0(T) - \frac{15}{2}\chi(T) \tag{3.2}$$

Moreover, equality holds if and only if $T \in H^4(1-\frac{1}{2}\chi(T))$, *where the class of triangulations* $H^4(n)$ *is defined inductively according to: (a) The boundary complex of any abstract five-simplex (*$\mathrm{Bd}\sigma$*) is a member of* $H^4(0)$. *(b) If* K *is in* $H^4(0)$ *and* σ *is a four-simplex of* K, *then* K' *is in* $H^4(0)$, *where* K' *is any complex obtained from* K *by deleting* σ *and adding the join of the boundary complex* $\mathrm{Bd}\sigma$ *and a new vertex distinct from the vertices of* K. *(c) If* K *is in* $H^d(n)$, *then* K' *is in* $H^d(n+1)$ *if there exist two four-simplexes* σ_1 *and* σ_2 *with no common vertices and a dimension preserving simplicial map* ϕ *from* $K-\sigma_1-\sigma_2$ *onto* K' *which identifies* $\mathrm{Bd}\sigma_1$ *with* $\mathrm{Bd}\sigma_2$ *but otherwise is one to one.*

In other words $H^4(0)$ is built up by gluing together five-dimensional simplexes through their four dimensional faces and considering only the boundary of this resulting complex. $H^4(n)$ differs from $H^4(0)$ by the fact that it has n handles. This way of constructing a triangulation of a four-sphere has a natural connection with the definition of a baby universe (see appendix B).

A baby universe is associated with a triangulation in which we can distinguish two pieces. A piece that contains the majority of the simplices of the triangulation that is called the "mother", and a small part called the "baby". In the "minbus" (minimun neck baby universes) the two parts are glued together along the boundary of a four dimensional simplex (in four dimensions) that is the "neck" of the

baby universes. Thus the "stacked spheres" can be considered as a network of minbus in which the mother is disappeared and the baby universes have a minimal volume, that is the boundary of a five simplices minus the simplices of the necks through which they are glued to the others . We will exploit this parallel in section 3.4 to give an estimate of the number of distinct stacked spheres.
In four-dimensions Dehn-Sommerville equations [60] are

$$\begin{aligned} N_0 - N_1 + N_2 - N_3 + N_4 &= \chi \\ 2N_1 - 3N_2 + 4N_3 - 5N_4 &= 0 \\ 2N_3 = 5N_4 \quad , \end{aligned} \tag{3.3}$$

substituting the third equation in the second we have

$$N_1 = \frac{3}{2}N_2 - \frac{5}{2}N_4 \quad , \tag{3.4}$$

and substituting these last equation in the first equation of 3.4 we get

$$N_0 - \frac{1}{2}N_2 + N_4 = \chi \quad . \tag{3.5}$$

This imply that

$$b \leq 5 + \frac{10\chi}{N_2} \quad , \tag{3.6}$$

and substituting 3.4 and 3.5 in 3.2 we obtain

$$b \geq 4 - \frac{10\chi}{N_2} \quad . \tag{3.7}$$

Thus in the limit of large N_2, $4 \leq b \leq 5$.

3.3 Canonical Partition Function

Now we introduce more extensively the results in four dimensions for the canonical partition function which has been rapidly summarized in section 2.2 of chapter 2. Recently it has been analytically shown [30] [92] that the dynamical triangulations in four dimensions are characterized by two phases: a strong coupling phase in the region $k_2^{inf} = \log 9/8 < k_2 < k_2^{crit}$ and a weak coupling phase for $k_2 > k_2^{crit}$. k_2^{crit} is the value of k_2 for which in the infinite volume limit the theory has a phase transition from the strong to the weak phase (for a detailed analysis see

[30]). The transition between these two phases is characterized by the fact that the sub dominant asymptotic of the number of distinct triangulations passes from an exponential to polynomial behaviour (c.f.[30]). Presently the precise value of k_2^{crit} has not been established yet, it is just known that it is close but distinct from $k_2^{max} = \log 4$, recent numerical simulations suggest that $k_2^{crit} = 1.24$.
In the strong coupling phase the leading term of the asymptotic expansion of the canonical partition function is (c.f. [30])

$$Z(N_4, k_2) = c_4 \left(\frac{A(k_2)+2}{3A(k_2)} \right)^{-4} \quad {N_4}^{-5} exp[-m(\eta^*) N^{\frac{1}{n_H}}] \\ exp\left[[10 \log \frac{A(k_2)+2}{3}] N_4 \right] \tag{3.8}$$

where for notational convenience we have set

$$\begin{aligned} A(k_2) \quad &\equiv \quad \left[\frac{27}{2} e^{k_2} + 1 + \sqrt{(\frac{27}{2} e^{k_2} + 1)^2 - 1} \right]^{1/3} \\ &+ \quad \left[\frac{27}{2} e^{k_2} + 1 - \sqrt{(\frac{27}{2} e^{k_2} + 1)^2 - 1} \right]^{1/3} - 1 \end{aligned} \tag{3.9}$$

and

$$\eta^*(k_2) = \frac{1}{3}(1 - \frac{1}{A(k_2)}) \quad . \tag{3.10}$$

The explicit form of $m(\eta^*(k_2))$ and n_H, the Hausdorff dimension, are at present unknown.

In the weak phase $k_2 > k_2^{crit}$ the number of distinct dynamical triangulations with equal curvature assignment have a power law behaviour in N_4. The phase is characterized by two distinct asymptotic regimes of the leading term of the canonical partition function: the critical coupling regime and the weak coupling regime. In the critical coupling regime $k_2^{crit} < k_2 < k_2^{max}$ we have

$$Z(N_4, k_2) \asymp c_4 \left(\frac{A(k_2)+2}{3A(k_2)} \right)^{-4} \quad {N_4}^{\tau(\eta^*)-5} \\ exp\left[[10 \log \frac{A(k_2)+2}{3}] N_4 \right] \quad , \tag{3.11}$$

in which $\tau(\eta^*)$ is not explicitly known.

Instead in the weak coupling regime $k_2 > k_2^{max}$

$$Z(N_4, k_2) \asymp \frac{c_4 e}{\sqrt{2\pi}} \frac{\eta_{max}^{-1/2}(1-2\eta_{max})^{-4}}{\sqrt{(1-3\eta_{max})(1-2\eta_{max})}} (\widehat{N_4+1})^{\tau-11/2} \cdot \frac{e^{(\widehat{N_4+1})f(\eta_{max},k_2)}}{k_2^{sup}-k_2} \tag{3.12}$$

in which $\eta^{max} = \frac{1}{4}$ for the sphere S^4,$\widehat{N_4+1} = 10(N_4+1)$ and

$$f(\eta, k_2) \equiv -\eta \log \eta + (1-2\eta)\log(1-2\eta) - (1-3\eta)\log(1-3\eta) + k_2\eta \quad . \tag{3.13}$$

From this form of the canonical partition function it follows that in the case of the sphere S^4 and in the infinite volume limit the average value of b, see equation (3.1), is a decreasing function of k_2 in the critical coupling regime and it is constantly equal to 4 in the weak coupling regime, that is to say

$$\begin{aligned} \lim_{N_4 \mapsto \infty} < b >_{N_4} &= \frac{1}{\eta(k_2)}, \quad k_2^{crit} \le k_2 \le k_2^{max} \\ \lim_{N_4 \to \infty} < b >_{N_4} &= 4, \quad k_2 \ge k_2^{max} \quad . \end{aligned} \tag{3.14}$$

If we look at the average curvature we have

$$\lim_{N_4 \to \infty} \frac{1}{N_4} < \sum_B K(B) vol_4(B) > = \pi a^2 \sqrt{3} \left(10\eta^*(k_2) - \frac{5}{\pi} \cos^{-1}\frac{1}{4} \right) \qquad k_2^{crit} < k_2 \le k_2^{max} \tag{3.15}$$

and

$$\lim_{N_4 \to \infty} \frac{1}{N_4} < \sum_B K(B) vol_4(B) > = \pi a^2 \sqrt{3} \left(\frac{5}{2} - \frac{5}{\pi} \cos^{-1}\frac{1}{4} \right) \qquad k_2 > k_2^{max} \quad , \tag{3.16}$$

in which $vol_4(B)$ is the volume of the four simplexes incident on the bone B, and it is $a^4 \frac{\sqrt{5}}{2^4 \cdot 6} q(B)$. As we have already said the value of the average curvature is saturated.

This result was already known in numerical simulations [59] and, as we will see in detail in the next section, it is due to the prevalence of stacked spheres in the weak coupling regime.

3.4 Elongated Phase

We have seen that in the case of triangulations of the sphere S^4 for $k_2 \mapsto \log 4$, in the infinite volume limit $< b >_{N_4} \mapsto 4$. Walkup's theorem [60] implies that the minimum value of b is reached on triangulations K of the sphere S^4 that belong to $H^4(0)$ (stacked spheres). Then for $k_2 > \log 4$ one has $< b >_{N_4} = 4$, that means that in this region of k_2 the statistical ensemble of quantum gravity is strongly dominated by stacked spheres.
As explicitly shown in [60], the elements of $H^4(0)$ can be put in correspondence with a tree structure. Let us recall that a $d-$dimensional simplicial complex T, $d \geq 1$, is called a simple $d-$tree if it is the closure of its $d-$simplexes $\sigma_1, ..., \sigma_t$ and these $d-$simplexes can be ordered in such a way that:

$$\mathrm{Cl}\ \sigma_j \bigcap \left\{ \bigcup_{i=1}^{j-1} \mathrm{Cl}\ \sigma_i \right\} = \mathrm{Cl}\ \tau_j \tag{3.17}$$

for some $(d-1)$-face τ_j of σ_j, $j \geq 2$, and where the τ_j are all distinct. This ordering of the simplexes of T induces a natural ordering of its vertices in $\nu_1, ..., \nu_{t+d}$, where ν_{i+d} is the vertex of σ_i not in $\mathrm{Cl}\ \tau_i$. Note that the interior part of T contains the simplexes σ_i and faces τ_i. The boundary of T, $\mathrm{Bd}\ T$, consists of the boundary of the σ_i minus the τ_i, and is topologically equivalent to S^{d-1}.
It can be shown[60], and it is very easy to check, that any element of $K \in H^d(0)$ is the boundary of a simple $(d+1)$-tree T, and that K determines uniquely the simple $(d+1)$-tree T for $d \geq 2$.

Note that any stacked sphere $K \in H^4(0)$ can be mapped into a tree graph (see also the reference [68]). This mapping is defined in the following way, let's consider the (unique) simple five-simple tree T associated with K, every five-simplex is mapped into a vertex and every four-dimensional face in common with two five simplexes is mapped into an edge which has endpoints at the two vertices which represent the two five simplexes (see fig. 3.1).
Since the map between K and T is one to one, we have a map from a stacked-sphere into a tree-graph whose number of links at every vertex can be at most 6 (since a five simplex has *six* faces). This map, from the stacked spheres $H^4(0)$ to the simple tree-graphs, is not one to one. The mathematical reason is that the previous construction maps every T to a tree-graph by an application that is the dual map restricted to the five and four dimensional simplexes of T in its domain and whose image is the tree graphs that are the 1-dimensional skeleton of the dual complex. It is well known that the dual map is a one to one correspondence only

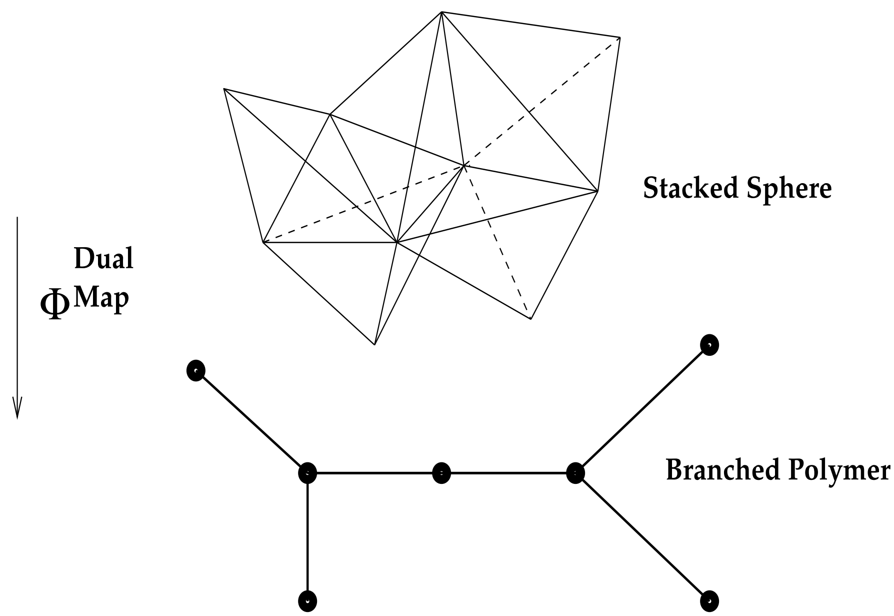

Figure 3.1: Two dimensional stacked sphere that is mapped into a branched polymer by the dual map Φ

if we take in account the simplexes of T of any dimension. It follows that our map is such that a simple tree-graph may correspond to many stacked spheres. To illustrate better this point we shall use a picture closer to the physical intuition. Let us consider a stacked sphere K and its simple tree T. Consider on T one of its ordered simplexes σ_j and let τ_j be the face that it shares with one of the simplexes $\sigma_1, \ldots, \sigma_t$ introduced previously. Repeating the same arguments used in the calculations of the inequivalent triangulations for baby universes [57] (see also appendix B), we can cut T in τ_j in two parts, such that each part has two copies of τ_j as a part of its boundary. Since τ_j is a four dimensional simplex it is easy to admit that we can glue this two different copies in $5 \cdot 4 \cdot 3$ ways to rebuild again a simple five tree T'. In general all the possible gluings will generate distinct triangulations and consequently distinct stacked spheres (if the two parts are highly symmetric triangulations some of the 60 ways of joining will not be distinct but, since for large N_4 asymmetrical triangulations will dominate, the number of case in which this will happen will be negligible). The corresponding tree graph associated with these configurations T' will always be the same since this operation has not modified the one skeleton of the tree T.

In statistical mechanics simple-tree structures correspond to "branched-polymers"

(see appendix A). So we have that for $k_2 \geq k_2^{max}$ the dominant configurations for the triangulations of S^4 are branched polymers. This fact was already observed in numerical simulations in four dimensions. In [61]a network of baby universes of minimum neck (minbu) and of minimum size (blips) without a mother universe was obtained. These have been interpreted as branched polymer like structures.

3.5 Stacked Spheres and Branched Polymers

In this section we will establish a parallel between the standard mean field theory of branched polymers and the stacked spheres, in the sense that we will use the counting techniques of branched polymers to give an upper and lower estimate of the number of inequivalent stacked spheres. As we have stressed in the previous section there exists a map between the stacked spheres and tree graphs and this map is *not* one to one in the sense that there are more stacked spheres than tree graphs. This means that the number of inequivalent stacked spheres with a fixed number N_5 of five-simplexes is bounded below by the corresponding number of tree graphs. Now we will study the statistical behaviour of the tree graphs using the measure of simplicial quantum gravity and restricting it to the stacked spheres. In this analysis we shall follow the theory of branched polymers as explained in appendix A. The reader may refer to it for the details. A more mathematical analysis on the enumeration of inequivalent tree graphs is contained in [62].

First of all we notice that the boundary of a five-simplex has six four-simplexes and every edge of a tree graph correspond to a cancellation of two four-simplexes in the corresponding boundary of the stacked sphere. Since in a tree graph with N_5 vertices there are $N_5 - 1$ edges, we have that the boundary of a stacked sphere, whose corresponding tree graph has N_5 points, is made by a number N_4 of four-simplexes

$$N_4 = 4N_5 + 2 \quad . \tag{3.18}$$

From the condition for stacked spheres (3.2), we have that the Einstein-Hilbert action, for dynamical triangulations, restricted to the stacked spheres is

$$S = N_4(k_4 - \frac{5}{2}k_2) - 5k_2 \quad . \tag{3.19}$$

It follows that the Gibbs factor for the ensemble of tree graphs (branched polymers) corresponding to stacked spheres is

$$exp\left(-(4N_5+2)(k_4-\frac{5}{2}k_2)+5k_2\right) \quad . \tag{3.20}$$

Following the same notation as in A.1 , let $r^6(N_5)$ be the number of inequivalent rooted tree graphs with N_5 vertices and with at most six incident edges on each vertex, with one rooted vertex with one incident edge. $\xi^6(N_5)$ is the number of inequivalent tree graphs with N_5 vertices. By A.3 we have

$$\xi^6(N_5) = \frac{1}{N_5} r^6(N_5+1) \tag{3.21}$$

Now let $R^6(k_4,k_2)$ be the partition function for rooted tree graphs with statistical weight (3.20), and $Z^6(k_4,k_2)$ be the partition function for unrooted tree graphs. We have

$$R^6(k_4,k_2) \equiv \sum_{N_5=2}^{\infty} e^{-(4(N_5-1)+2)(k_4-\frac{5}{2}k_2)+5k_2} r^6(N_5) \tag{3.22}$$

$$Z^6(k_4,k_2) \equiv \sum_{N_5=1}^{\infty} e^{-(4N_5+2)(k_4-\frac{5}{2}k_2)+5k_2} \xi^6(N_5) \tag{3.23}$$

Let's define

$$R^{*6}(\triangle k_4) \equiv \left(e^{-2(k_4-5)}R^6(k_4,k_2)\right) = \sum_{N_5=2}^{\infty} e^{-4(N_5-1)\triangle k_4} r^6(N_5) \tag{3.24}$$

$$Z^{*6}(\triangle k_4) \equiv \left(e^{-2(k_4-5)}Z^6(k_4,k_2)\right) = \sum_{N_5=1}^{\infty} e^{-4N_5\triangle k_4} \xi^6(N_5) \quad , \tag{3.25}$$

where $\triangle k_4 \equiv k_4 - \frac{5}{2}k_2$. It is easy to see, from 3.21, that

$$R^{*6}(\triangle k_4) = -\frac{1}{4}\frac{d}{d\triangle k_4} Z^{*6}(\triangle k_4) \quad . \tag{3.26}$$

From A.11 we know that the asymptotic behavior of the susceptibility $\chi(k_4,k_2)$ is given by

$$\chi(k_4,k_2) \asymp \frac{\partial^2}{\partial k_4^2} Z(k_4,k_2) \tag{3.27}$$

Then

$$\chi(\triangle k_4) \asymp \frac{d^2}{d(\triangle k_4)^2} Z^{*6}(\triangle k_4) \asymp \frac{d}{d(\triangle k_4)} R^{*6}(\triangle k_4) \tag{3.28}$$

Now we shall study the critical behaviour of this system in order to obtain some information about the critical behaviour of the stacked spheres.
From the identity A.12 we have that

$$R^{*6}(\triangle k_4) = e^{-4\triangle k_4} \left(\sum_{\gamma=0}^{5} \frac{1}{\gamma!} [R^{*6}(\triangle k_4)]^{\gamma} \right) \quad . \tag{3.29}$$

This last equation suggest to define a function $F(R^{*6}, \triangle k_4)$ of R^{*6} and $\triangle k_4$ and to apply the implicit function theorem to it in
order to obtain information on $\triangle k_4$ as a function of R^{*6}.
More precisely the following function $F(R^{*6}, \triangle k_4)$ of R^{*6} and $\triangle k_4$

$$F(R^{*6}, \triangle k_4) \equiv R^{*6} - e^{-4\triangle k_4} \left(\sum_{\gamma=0}^{5} \frac{1}{\gamma!} [R^{*6}]^{\gamma} \right) \tag{3.30}$$

at the point $\left((R^{*6})^0, (\triangle k_4)^0\right)$ where

$$F((R^{*6})^0, (\triangle k_4)^0) = 0 \quad , \tag{3.31}$$

if it is true the following condition

$$\left.\frac{\partial F}{\partial \triangle k_4}\right|_{((R^{*6})^0, (\triangle k_4)^0)} \quad , \tag{3.32}$$

defines by the implicit function theorem, locally, $\triangle k_4$ as
function of R^{*6}.
Differentiating equation 3.29 we get a differential equation for $R^{*6}(\triangle k_4)$

$$\frac{d}{d\triangle k_4} R^{*6}(\triangle k_4) = -4R^{*6}(\triangle k_4) \left[1 + \frac{e^{-4\triangle k_4}}{5!} (R^{*6}(\triangle k_4))^5 - R^{*6}(\triangle k_4) \right]^{-1} \tag{3.33}$$

So $R^{*6}(\triangle k_4)$ shows a singularity when

$$R^{*6}\left((\triangle k_4)^c\right) = 1 + \frac{e^{-4(\triangle k_4)^c}}{5!} \left(R^{*6}((\triangle k_4)^c) \right)^5 \tag{3.34}$$

This equation has only one positive solution and since $\frac{d}{d\triangle k_4} R^{*6}|_{(\triangle k_4)^c}$ diverges, the inverse function tends to zero

$$\left.\frac{d(\triangle k_4)(R^{*6})}{dR^{*6}}\right|_{(\triangle k_4)^c} \mapsto 0 \tag{3.35}$$

Observe that $F(R^{*6}, \triangle k_4)$ is an analytic function, and in the point $\left((R^{*6})^c, (\triangle k_4)^c\right)$ fixed by equation 3.34, we have

$$\left.\frac{\partial F}{\partial \triangle k_4}\right|_{((R^{*6})^c,(\triangle k_4)^c)} = 4e^{-4(\triangle k_4)^c}\left(\sum_{\gamma=0}^{5}\frac{1}{\gamma!}[(R^{*6})^c]^\gamma\right) > 0 \tag{3.36}$$

and pairwise

$$\left.\frac{\partial F}{\partial R^{*6}}\right|_{((R^{*6})^c,(\triangle k_4)^c)} = 1 - e^{-4(\triangle k_4)^c}\left(\sum_{\gamma=0}^{5}\frac{1}{\gamma!}[(R^{*6})^c]^\gamma\right) = 0 \quad , \tag{3.37}$$

this last equation explains why we cannot express in the neighbourhood of $\left((R^{*6})^c, (\triangle k_4)^c\right)$ R^{*6} as function of $\triangle k_4$ by applying the implicit function theorem to 3.30. These arguments are enough to say that in the neighbourhood of $\left((R^{*6})^c, (\triangle k_4)^c\right)$ $(\triangle k_4)(R^{*6})$ is an analytic function of R^{*6}. Let us fix the following notation

$$F_{R^c} \equiv \left.\frac{\partial F}{\partial R^{*6}}\right|_{((R^{*6})^c,(\triangle k_4)^c)} \qquad F_{\triangle k_4^c} \equiv \left.\frac{\partial F}{\partial \triangle k_4}\right|_{((R^{*6})^c,(\triangle k_4)^c)} \tag{3.38}$$

$$F_{R^c R^c} \equiv \left.\frac{\partial^2 F}{\partial (R^{*6})^2}\right|_{((R^{*6})^c,(\triangle k_4)^c)}$$

and so on. It is straightforward to calculate that

$$F_{R^c R^c} = -e^{-4(\triangle k_4)^c}\left(\sum_{\gamma=0}^{3}\frac{1}{\gamma!}[(R^{*6})^c]^\gamma\right) < 0 \quad . \tag{3.39}$$

Applying again the implicit function theorem we obtain

$$\left.\frac{d^2(\triangle k_4)(R^{*6})}{d(R^{*6})^2}\right|_{(\triangle k_4)^c} = -\frac{F_{R^c R^c}}{F_{\triangle k_4^c}} > 0 \quad . \tag{3.40}$$

All these facts allow to write the following expansion

$$\begin{aligned}\triangle k_4(R^{*6}) - \triangle k_4{}^c &= \frac{1}{2}\left.\frac{d^2(\triangle k_4)(R^{*6})}{d(R^{*6})^2}\right|_{R^{*6}((\triangle k_4)^c)}\left(R^{*6}(\triangle k_4) - R^{*6}(\triangle k_4{}^c)\right)^2 \\ &\quad + o\left(\left(R^{*6}(\triangle k_4) - R^{*6}(\triangle k_4{}^c)\right)^2\right) \quad .\end{aligned} \tag{3.41}$$

So near $(\triangle k_4)^c$ we can write

$$R^{*6}(\triangle k_4) \asymp R^{*6}((\triangle k_4)^c) + C\left((\triangle k_4) - (\triangle k_4)^c\right)^{\frac{1}{2}} \tag{3.42}$$

From (3.28) we get

$$\chi^6(\triangle k_4) \asymp \left((\triangle k_4) - (\triangle k_4)^c\right)^{-\frac{1}{2}} \tag{3.43}$$

This means that the critical exponent of the susceptibility for the system of these tree graphs is $\gamma = \frac{1}{2}$.

The partition function that has been studied is a lower estimate of the partition function of the stacked spheres. Consider, now, a stacked sphere K and a face τ_j through which two five-simplexes are glued together (a link on the corresponding tree graph). We can glue two four face of a stacked sphere in 5·4·3 different ways, for large number of simplexes this will generate distinct configurations of stacked spheres whose associated tree-graph is always the same. Repeating the same argument for every j, $j = 1, \ldots, N_5 - 1$, (that is to say for every link of the dual tree graph) we obtain a factor, $(5\cdot 4\cdot 3)^{N_5-1}$, that multiplied by $r^6(N_5)$ and $\xi^6(N_5)$, gives an upper bound on the number of , respectively, rooted and unrooted inequivalent stacked spheres. In other words

$$Z_{tree}(k_4, k_2) \leq Z_{s.s.}(k_4, k_2) \leq \tilde{Z}_{tree}(k_4, k_2) \tag{3.44}$$

where $Z_{tree}(k_4, k_2)$ is the partition function for tree graphs studied above, $Z_{s.s.}(k_4, k_2)$ is the partition function for stacked spheres and $\tilde{Z}_{tree}(k_4, k_2)$ is the partition function for tree graphs with the additional weight defined above.
A similar analysis as for $Z_{tree}(k_4, k_2)$ shows that the critical line of $\tilde{Z}_{tree}(k_4, k_2)$ is, of course, a straight line parallel and above respect to the k_4 axis to $Z_{tree}(k_4, k_2)$ one, and the susceptibility exponent is again $\gamma = \frac{1}{2}$.
Obviously the estimates (3.44) are true for the canonical partition functions too,

$$Z_{tree}(N_4, k_2) \leq Z_{s.s.}(N_4, k_2) \leq \tilde{Z}_{tree}(N_4, k_2) \quad , \tag{3.45}$$

and the previous calculations show that

$$Z_{tree}(N_4, k_2) \asymp N_4^{-\frac{5}{2}} e^{-N_4(k_4 - \frac{5}{2}k_2 - t_4)} \tag{3.46}$$

and

$$\tilde{Z}_{tree}(N_4, k_2) \asymp N_4^{-\frac{5}{2}} e^{-N_4(k_4 - \frac{5}{2}k_2 + t_4 - \frac{1}{4}\log 60)} \quad , \tag{3.47}$$

where t_4 is a constant that may be calculated by the equation 3.34. More easily from the table in reference [62] we get that $t_4 \approx \frac{1}{4}\log 0.34$.
The circumstance that in the weak coupling region the partition function of quantum gravity, as found in an analytically way in [30], is strongly dominated by stacked spheres, allows us to write

$$Z_{s.s.} \asymp N_4^{\gamma_s - 3} e^{N_4 k_4^c(k_2)} \quad , \tag{3.48}$$

where γ_s is the susceptibility exponent [52].
Thus the critical line of the stacked spheres is a straight line parallel and among the critical lines of the systems of the two branched polymers. This implies

$$k_4 - \frac{5}{2}k_2 + t_4 \leq k_4^c(k_2) \leq k_4 - \frac{5}{2}k_2 + t_4 - \frac{1}{4}\log 60 \tag{3.49}$$

Moreover from the equations 3.46 and 3.47 the one loop green functions [64] of the two model of branched polymers have respectively , near their critical lines , the asymptotic behaviour

$$G_{tree}(\triangle k_4) \asymp \mathrm{cost}_1 + (\triangle k_4 - \triangle k_4{}^c)^{\frac{1}{2}} \qquad \tilde{G}_{tree}(\triangle \tilde{k}_4) \asymp \mathrm{cost}_2 + (\triangle \tilde{k}_4 - \triangle \tilde{k}_4{}^c)^{\frac{1}{2}} \tag{3.50}$$

This last equation together the equations 3.45 and 3.48 prove that the one loop function of the stacked spheres $G_{s.s.}(k_4, k_2)$ near the critical line has the asymptotic behaviour

$$G_{s.s.}(k_4, k_2) \asymp \mathrm{cost}_3 + (k_4 - k_4^c(k_2))^{1-\gamma_s} \quad , \tag{3.51}$$

with $\gamma_s < 1$ (the value $\gamma_s = 1$ is not allowed because in this case the one loop green function of stacked spheres at the critical line would have a behaviour like $\sum_{N_4=5}^{\infty} 1/N_4$ that is divergent and then incompatible with the upper bound given by the second equation of 3.50).

Motivated by physical considerations, we can use a well known argument [64] [67] in favor of the fact that the susceptibility exponent of the stacked spheres is $\gamma_s = \frac{1}{2}$. More precisely we will show that a model of proliferating baby universes, with the measure of quantum gravity restricted to stacked spheres, can be put in correspondence with the statistical system of stacked spheres.
Let us consider four dimensional triangulations that are (boundary of the) stacked spheres in which there can be loops made by two three-dimensional simplexes. This is possible whenever the stacked spheres are pinched on a three simplex

creating a bottle neck loop of two three-simplexes. These loops could be either the loops of a Green function or the minimum bottle neck of a baby universes. This class of triangulations, following the notation in literature, is called $\mathbf{T}_2$. The other class of triangulations is the stacked spheres in which the minimum loop length can be made by the boundary of a four-simplex that are five three-simplexes. We call this last class $\mathbf{T}_5$.

In the two dimensional theory, the introduction in $\mathbf{T}_2$ of two-link loops (the two dimensional analogous of two three-simplex loop) corresponds in the matrix model ϕ^3 to consider Feynman diagrams with self-energy (c.f. [64] [67]).
Since the minima loops of $\mathbf{T}_2$ and $\mathbf{T}_5$, for which they differ, are of the order of lattice spacing we will expect that the two classes of triangulations, as a statistical mechanics system, coincide in the scaling limit, that is to say they belong to the same universality class.
Let's consider the minimum neck one loop function $G(\triangle k_4)$ in $\mathbf{T}_2$. In every triangulation of $\mathbf{T}_2$ we can cut out the maximal size baby universe of minimum neck and close the two three-simplex loop. We will obtain again a triangulation that belongs to $\mathbf{T}_2$. In this way we will obtain all the triangulations of $\mathbf{T}_2$ from the triangulations of the stacked spheres $\mathbf{T}_5$ considering that for each three-simplex either leave them in their actual form or we can open the triangulation to create a two three-simplex loop and gluing on it a whole one loop universe $G(\triangle k_4)$. We note that in the triangulations of $\mathbf{T}_5$ $\overline{N}_3 = 5/2\overline{N}_4$ (Dehn-Sommerville). Calling the one loop function of $\mathbf{T}_5$ $\overline{G}(\overline{\triangle k_4})$, the above considerations lead to the identity

$$G(\triangle k_4) = \sum_{\mathbf{T}\in\mathbf{T}_5} e^{-\overline{N}_4 \triangle k_4} (1 + G(\triangle k_4))^{\overline{N}_3} = \sum_{\mathbf{T}\in\mathbf{T}_5} e^{-\overline{N}_4 \overline{\triangle k_4}} = \overline{G}(\overline{\triangle k_4}) \quad , \qquad (3.52)$$

in which $\triangle k_4 = k_4 - 5/2 k_2$ comes out from restricting the action of quantum gravity to stacked spheres 3.19, and where we have defined

$$\overline{\triangle k_4} = \triangle k_4 - \frac{5}{2} \log\left(1 + G(\triangle k_4)\right) \quad . \qquad (3.53)$$

By 3.52 we can also write last equation as

$$\triangle k_4 = \overline{\triangle k_4} + \frac{5}{2} \log\left(1 + \overline{G}(\overline{\triangle k_4})\right) \quad . \qquad (3.54)$$

By universality and the estimates (3.45) it follows that near the critical point we have that $G(\triangle k_4) \asymp \text{cost} + (\triangle k_4 - \triangle k_4^c)^{1-\gamma_s}$ with $\gamma_s < 1$.

The susceptibility functions of $\mathbf{T_2}$ and $\mathbf{T}_5$ by 3.52 are

$$\chi(\triangle k_4) \asymp -\frac{d}{d\triangle k_4} G(\triangle k_4) \quad \overline{\chi}(\overline{\triangle k_4}) \asymp -\frac{d}{d\overline{\triangle k_4}} \overline{G}(\overline{\triangle k_4}) \tag{3.55}$$

From 3.54 we have

$$\frac{d(\triangle k_4)}{d(\overline{\triangle k_4})} = 1 - \frac{5}{2} \frac{\overline{\chi}(\overline{\triangle k_4})}{(1+\overline{G}(\overline{\triangle k_4})} \quad . \tag{3.56}$$

If we calculate di derivative with respect to $\triangle k_4$ of the one loop function $G(\triangle k_4)$ and use the previous equation we get

$$\chi(\triangle k_4) = \frac{\overline{\chi}(\overline{\triangle k_4})}{1 - \frac{5}{2} \frac{\overline{\chi}(\overline{\triangle k_4})}{(1+\overline{G}(\overline{\triangle k_4})}} \quad . \tag{3.57}$$

Now it is clear that $\chi(\triangle k_4) \mapsto +\infty$ for $\triangle k_4 \mapsto (\triangle k_4)^c$ and with the same critical exponent $\overline{\chi}(\overline{\triangle k_4}) \mapsto +\infty$ for $\overline{\triangle k_4} \mapsto (\overline{\triangle k_4})^c$. From equation (3.57) when $\chi(\triangle k_4) \mapsto +\infty$ we have that $\overline{\chi}\left(\overline{\triangle k_4}\right) \mapsto \frac{2}{5}\left(1 + \overline{G}\left(\overline{\triangle k_4}(\triangle {k_4}^c)\right)\right) < +\infty$, then the system $\mathbf{T}_5$ is above his critical line, i.e. $\overline{\triangle k_4}(\triangle {k_4}^c) > \overline{\triangle k_4}^c$. These facts imply that at $\overline{\triangle k_4}(\triangle {k_4}^c)$ $d(\triangle k_4)/d(\overline{\triangle k_4}) = 0$ and around it $\overline{\chi}(\overline{\triangle k_4})/(1+\overline{G}(\overline{\triangle k_4}))$ is a decreasing monotonic function by equation 3.56 because $\chi(\triangle k_4) \mapsto +\infty$, we can expand equation 3.54 and 3.57 around $\overline{\triangle k_4}(\triangle {k_4}^c)$

$$\triangle k_4 - \triangle {k_4}^c = \text{cost} \left(\overline{\triangle k_4} - \overline{\triangle k_4}(\triangle {k_4}^c)\right)^2 \quad . \tag{3.58}$$

$$\chi(\triangle k_4) \asymp \frac{1}{\overline{\triangle k_4} - \overline{\triangle k_4}(\triangle {k_4}^c)} \asymp \frac{1}{\sqrt{\triangle k_4 - \triangle {k_4}^c}} \quad . \tag{3.59}$$

The second asymptotic equality of the last equation implies

$$\gamma_s = \frac{1}{2} \tag{3.60}$$

The dominance of stacked spheres in the weak phase allows us to fix the parameter τ in the partition function of quantum gravity in the weak coupling regime.

$$\tau - \frac{11}{2} = -\frac{5}{2} \Longrightarrow \tau = 3 \tag{3.61}$$

These considerations put evidence that the branched polymer phase of 4-D Dynamical Triangulations is a mean field phase.

Chapter 4

Lattice Gauge Theory of Gravity

4.1 Introduction

The model of dynamical triangulations, that we have analysed up to now, has been very popular in recent years. The reason of this is due to the excellent results in two dimensions where the critical exponents coincide perfectly with the critical exponents of two dimensional continuum quantum gravity. This has encouraged people to explore

the model in four dimensions. The main aim has been to explore the dynamics of this discrete theory in order to define a four-dimensional quantum theory of gravity at the critical point (if any) of the parameters space. At the beginnings a second order phase transition has been observed by numerical simulations [52]. Successive and more accurate simulations [53] [54] have shown that the phase transition is of first order.

At the moment the situation is not completely clear and further investigations seem necessary, but the general agreement of the scientific community about the first order nature of the phase transition points out that it is necessary to find a new lattice theory of gravity which fits with the general requirements of Euclidean lattice field theory. The following model is an attempt in this direction and might be an answer to this request. It has been proposed originally and independently by different authors [8] [72] [79], and the original part of the content of this chapter, based on the work [91], is a continuation and an improvement of these attempts, in particular of [72] (connected to this work are [73] [74]). We begin in section 4.2 by reviewing some concepts contained in [72] and in particular we show how to associate to each dual Voronoï edge (see appendix C and references [18] [19] and [20] p. 395) of the original n-dimensional simplicial complex a Poincarè transfor-

mation by fixing an orthonormal reference frame in each n-dimensional simplex. Then we recall the group theoretical action which is a functional of the sine of the deficit angles. Respect to the original work [72] it is shown that the action is independent from the orientation of the hinges and from the starting n-simplex in which it is written .

In section 4.3 following [72] we write the group theoretical action

completely defined on the dual Voronoï complex of the original simplicial complex. Two sets of variables play an important role: the connection matrices $U^a_b(\alpha,\beta)$ and the normals $b^a_{\alpha\beta}(\alpha)$ to the $n-1$ faces of each n-simplex α. In the second order formalism the connection matrices $U^a_b(\alpha,\beta)$ are functions of the $b^a_{\alpha\beta}(\alpha)$. As anticipated in [72] we follow a first order formalism in which we consider the $U^a_b(\alpha,\beta)$ and the $b^a_{\alpha\beta}(\alpha)$ as independent variables. A first order formalism for Regge calculus has been done by J. Barrett [77].

We impose that the first order field variables satisfy only the two constraints 4.16 and 4.17. Finally, respect to [72], in the first order formalism we define a modified action in such a way that it is dependent only from each plaquette of the dual Voronoï complex .

In section 4.4 it is proved that the modified action is independent from the orientation of each plaquette.

In section 4.5 the field equations for the connection matrices $U^a_b(\alpha,\beta)$ are derived in the first order formalism in the approximation of "small deficit angles". It is proved that Regge calculus is a solution for these equations. This result is no longer obvious in the first order formalism when we consider the connection matrices independent from the $b^a_{\alpha\beta}(\alpha)$.

The equivalence of the first order formalism to the (second order) Regge calculus is the main new result in this chapter.

In section 4.6 we obtain the general field equations on lattice. These equations are divided in two sets. The set of equations for the connection matrices $U^a_b(\alpha,\beta)$ obtained by requiring that the action is stationary under the variation of the $U^a_b(\alpha,\beta)$, and the set of equations for $b^a_{\alpha\beta}(\alpha)$. These two sets of equations are obtained by using the method of the Lagrange multipliers through which we take into account the two sets of constraints, one set on the linear dependence among the $b^a_{\alpha\beta}(\alpha)$ of the same simplex α and the second set on the condition that the $b^a_{\alpha\beta}(\alpha)$ and $b^b_{\beta\alpha}(\beta)$, as the expressions of the same normal to the $n-1$ face seen, respectively, in the orthonormal reference frame of α and β, are connected by the matrix $U^a_b(\alpha,\beta)$. At the end of section 4.6 we introduce a method to calculate the Lagrange multipliers too.

In section 4.7 we define a path-integral for this classical theory of gravity on lattice and we prove that this quantum measure is invariant under local transformation of $SO(n)$.

Finally in section 4.8 we show as it is possible to define in this framework a coupling with fermionic matter following, as usual in lattice field theory, the example of the continuum theory.

The main feature of this model is that it is more general than Regge calculus and dynamical triangulations. The introduction of a reference frame in each n-dimensional simplex is the key concept for which this version of discrete gravity is different from the previous one. This is a way to translate on lattice the general principle of local invariance under a general Lie group which implies a structure , in the continuum, of a Principal Fiber Bundle with the fibers isomorphic to a Lie group (gauge theories) or to orthonormal reference frames connected among them by matrices of $SO(n)$ (General Relativity in the Riemmanian version). In this scheme we can naturally write on lattice the coupling of gravity with other fundamental forces and consider extensions to more general theories than gravity like, for example, supergravity (see for example [75] and [?]).

4.2 Group Action for Simplicial Gravity

In this section we shall introduce a group action for simplicial gravity [72] on the dual Voronoï complex [78] (see appendix C for a definition of Voronoï cells) of the original simplicial complex. In particular we will see that it is possible to associate to each dual edge $(\alpha, \alpha+1)$ a "Poincaré" transformation $U(\alpha, \alpha+1)$ as in lattice gauge theories. Of course by a "Poincaré" transformation from now on we mean in general a $SO(n)$ rotation and a translation.

We are now going to introduce a reference frame in each n-dimensional simplex and we shall see how these reference frames can be connected by using coordinates of the vertices of the common $n-1$-dimensional faces and a notion of Levi-Civita connection on simplicial manifolds. Consider an hinge h and let $\{P_1, ..., P_{n-1}\}$ its vertices. Suppose that this hinge is shared by N n-simplices $\{S_1, ..., S_N\}$ whose vertices are labelled in this way

$$S_\alpha \equiv \{P_1, ..., P_{n-1}, Q_{\alpha-1,\alpha}, Q_{\alpha,\alpha+1}\} \quad (\alpha = 1, ..., N) \quad . \tag{4.1}$$

In each simplex S_α we can fix an origin and a reference frame. In this frame the vertices of the simplex S_α have the following coordinates

$$
\begin{aligned}
P_i &\equiv \{y_i^a(\alpha)\} \quad a = 1, ..., n\ ,\ i = 1, ..., n-1 \\
Q_{\alpha-1,\alpha} &\equiv \{z^a_{\alpha-1,\alpha}(\alpha)\} \\
Q_{\alpha,\alpha+1} &\equiv \{z^a_{\alpha,\alpha+1}(\alpha)\}
\end{aligned}
\tag{4.2}
$$

Let's indicate by D_α the circumcenter of the simplex α and let $x^a(\alpha)$ their coordinates.
Now it is possible to associate uniquely to a dual edge an element of the Poincaré group $U(\alpha, \alpha+1) \equiv \{U^a_b(\alpha, \alpha+1), U^a(\alpha, \alpha+1)\}$ by requiring that

$$
\begin{aligned}
U^a_b(\alpha, \alpha+1) y^b_i(\alpha+1) + U^a(\alpha, \alpha+1) &= y^a_i(\alpha) \\
U^a_b(\alpha, \alpha+1) z^b_{\alpha,\alpha+1}(\alpha+1) + U^a(\alpha, \alpha+1) &= z^a_{\alpha,\alpha+1}(\alpha) \quad ,
\end{aligned}
\tag{4.3}
$$

in other words we are embedding α and $\alpha+1$ in $\mathbb{R}^n$ and adopting the standard notion of parallel displacement in $\mathbb{R}^n$ and we transport the origin of the reference frame of $\alpha+1$ from $\alpha+1$ to the origin of the orthonormal reference frame of α [8] [72]. This operation makes the position vectors in $\alpha+1$ of the vertices of the common face $S_\alpha \cap S_{\alpha+1}$ coincident with the position vectors of the same vertices in α. It follows that the matrix $U^a_b(\alpha, \alpha+1)$ is an orthogonal matrix which describes the change from the reference frame of $\alpha+1$ to α, considered now as two different reference frame of the same vector space ([8]). Since the simplicial manifold is assumed orientable [4] we can choose the reference frame in α and $\alpha+1$ in such a way that $U^a_b(\alpha, \alpha+1)$ are

$SO(n)$ matrices.

The family of all these matrices determine a unique connection that is the *Levi Civita or Regge connection.*

The arbitrariness of the choice of the reference frame in each simplex α implies a sort of local gauge invariance under Poincaré transformations $\Lambda(\alpha) = \{\Lambda^a_b(\alpha), \Lambda^a(\alpha)\}$:

$$
U(\alpha, \alpha+1) \mapsto \Lambda(\alpha) U(\alpha, \alpha+1) \Lambda^{-1}(\alpha+1) \quad ,
\tag{4.4}
$$

in particular the coordinates of the dual vertices (the circumcenters of the simplices S_α) transform as $x^a(\alpha) \mapsto \Lambda^a_b(\alpha) x^b(\alpha) + \Lambda^a(\alpha)$. Anyway from now on we decide to put the origin of the reference frames in the circumcenters, in such a way we always have $x^a(\alpha) = 0$.

To the hinge h it corresponds the dual two-dimensional plaquette that we still label by h. Now consider the following plaquette variable

$$U^{(h)}_{\alpha\alpha} = U(\alpha, \alpha+1)U(\alpha+1, \alpha+2)...U(\alpha-1, \alpha) \quad . \tag{4.5}$$

It is evident that $U^{(h)}_{\alpha\alpha}$ leaves the coordinates of the vertices of the hinge h unchanged and its translational part is zero. So it is a rotation of an angle $\theta(h)$ in the two-dimensional plain orthogonal to the hinge. It is easy to see [82] that the angle $\theta(h)$ is the deficit angle [1]

$$\theta(h) = 2\pi - \sum_{\alpha=1}^{N} \theta(\alpha, h) \quad , \tag{4.6}$$

where $\theta(\alpha, h)$ are the dihedral angle of the simplices S_α with the hinge h.

Let's choose $n-2$ linear independent edge vectors $E^a_1 = y^a_1 - y^a_{n-1}, ...E^a_{n-2} = y^a_{n-2} - y^a_{n-1}$ belonging to h , and consider the oriented volume $\mathcal{V}^{(h)\ ab}(\alpha)$ of h

$$\mathcal{V}^{(h)\ ab}(\alpha) =\equiv \frac{1}{(n-2)!} \epsilon^{ab}_{c_1...c_{n-2}} E^{c_1}_1(\alpha)...E^{c_{n-2}}_{n-2}(\alpha) \quad . \tag{4.7}$$

At this point it seems natural to propose the following gravitational action

$$I = -\frac{1}{2} \sum_h \left(U^{a_1 a_2\ (h)}_{\alpha\alpha} \mathcal{V}^{(h)}{}_{a_1 a_2}(\alpha) \right) \tag{4.8}$$

where $U^{a_1 a_2\ (h)}_{\alpha\alpha}$ are the elements of the orthogonal matrix 4.5. If we choose a reference frame in which the first two axis, 1 and 2, are on the two-dimensional plane orthogonal to the hinge h the matrix $U^{a_1 a_2\ (h)}_{\alpha\alpha}$ will be diagonal, with the diagonal elements equal to 1, except in the intersection of the first two rows and columns where it is

$$\begin{pmatrix} \cos\theta(h) & -\sin\theta(h) \\ \sin\theta(h) & \cos\theta(h) \end{pmatrix} \quad . \tag{4.9}$$

Pairwise in this reference frame the only non-zero components of the antisymmetric tensor $\mathcal{V}^{(h)\ ab}(\alpha)$ are $\mathcal{V}^{(h)\ 12}(\alpha)$ and $\mathcal{V}^{(h)\ 21}(\alpha)$, because in this reference frame the $n-2$ independent vectors of h have zero components on the first two axes. In particular the component $\mathcal{V}^{(h)\ 12}(\alpha)$ of the antisymmetric tensor is equal to $V(h)$ by the definition of the volume $V(h)$ of the hinge h. Since the trace in 4.8 is independent from the orthogonal reference frame chosen in the n-simplex α, the action 4.8 is equal to

$$I = \sum_h \sin\theta(h)\, V(h) \quad , \tag{4.10}$$

that for small deficit angles $\theta(h)$ reduces to Regge action [1].
Notice that in the definition 4.5 we have chosen the sequence of simplices $\alpha, \alpha+1, ..., \alpha-1, \alpha$. This sequence induces a direction of rotation in the two-dimensional plane orthogonal to the hinge h determined by the rotation matrix 4.5. We have implicitly assumed that this direction is the same direction generated by the ordered sequence of axes $\{1,2\}$ in the rigid rotation from the axis 1 to the axis 2.
Since the simplicial manifolds under consideration are assumed orientable [4], we can choose the same orientation in each n-dimensional simplex [6]. This orientation induces an orientation on each face but no orientation on the hinge because the two faces of the simplex, that share the hinge, induce opposite orientations on it. It follows that in principle there is no preferred orientation on the two-dimensional plane orthogonal to the hinge. So we can pairwise choose the sequence $\alpha, \alpha-1, ..., \alpha+1, \alpha$, the rotation matrix will be the transposed of 4.5 and the direction of rotation is now from the axis 2 to the axis 1, that is to say the opposite of the previous one. In other words respect to the first orientation, now the axis $1'$ coincides with 2 and $2'$ with 1. It follows that this change of orientation implies that $\epsilon^{a'_1 a'_2 ... a'_n} = -\epsilon^{a_1 a_2 ... a_n}$ so that $\mathcal{V}^{(h)\ 1'2'}(\alpha) = \mathcal{V}^{(h)\ 21}(\alpha)$.

Bearing in mind that the analogous of 4.9 with this new orientation is equal to its transposed , we have that the action now 4.8 is equal again to 4.10. In this way we have proved that the action 4.8 is independent from the chosen orientation.
The action 4.8 by construction is invariant under local Poincaré transformations (gauge invariance) and is independent from the starting simplex α. This last remark may be done more clear by the following lemma

Lemma 4.2.1 *The action (4.8) does not depend from the reference frame where it is written, in the sense that if α and δ_i are two simplices which have in common the same hinge h then*

$$U_{\alpha\alpha}^{a_1 a_2\ (h)} \mathcal{V}^{(h)}{}_{a_1 a_2}(\alpha) = U_{\delta_i \delta_i}^{a_1 a_2\ (h)} \mathcal{V}^{(h)}{}_{a_1 a_2}(\delta_i) \tag{4.11}$$

Dim : Let $\{\alpha...\delta_i\}$ the simplicial path from the simplex α to δ_i such that the simplices of the path share the hinge h. We have that from the definition of Levi-Civita connection

$$y_j^a(\alpha) - y_{n-1}^a(\alpha) = (U(\alpha, \alpha+1)...U(\delta_{i-1}, \delta_i))_b^a \left(y_j^b(\delta_i) - y_{n-1}^b(\delta_i)\right) \tag{4.12}$$

So for $SO(n)$ connection matrices is true the following

$$\begin{aligned}\mathcal{V}^{(h)}{}_{a_1 a_2}(\alpha) &= (U(\alpha,\alpha+1)...U(\delta_{i-1},\delta_i))_{a_1}^{a_1'} \\ &\quad\times\mathcal{V}^{(h)}{}_{a_1' a_2'}(\delta_i)\,(U(\delta_i,\delta_{i-1})...U(\alpha+1,\alpha))_{a_2}^{a_2'}\end{aligned}\tag{4.13}$$

By substituting this last equation in

$$U_{\alpha\alpha}^{a_1 a_2\ (h)}\mathcal{V}^{(h)}{}_{a_1 a_2}(\alpha)\tag{4.14}$$

we get the equality (4.11).

It is interesting to note that this action is similar to Wilson action [84] for lattice gauge theory (for basic notions of lattice field theory see [28] chap.6). Anyway in the form 4.8 the action is in a second order formalism since the plaquette variables $W_\alpha^{a_1 a_2}(h)$ are complicated functions by 4.3 of the coordinates of the vertices. Moreover the field variables belong both to the original triangulation and its Voronoï dual. We are now going to introduce a formalism that is written completely on the Voronoï complex and is a first order formalism like the Palatini version [85] of general relativity.

4.3 Action on the Dual Voronoï Complex

Let's consider the $n-1$-dimensional face $f_{\alpha\beta} \equiv \{P_1, ..., P_n\}$ between the n-dimensional simplices α and β. We label the coordinates of the vertices of the face $f_{\alpha\beta}$ by $y_i^a(\alpha)$ in the reference frame α. We define the vector $b_{\alpha\beta}^a$

$$\begin{aligned}b_{\alpha\beta}^a(\alpha) &= \epsilon_{ab_1...b_{n-1}}(y_1(\alpha)-y_n(\alpha))^{b_1}...(y_{n-1}(\alpha)-y_n(\alpha))^{b_{n-1}} \\ &= \epsilon_{ab_1...b_{n-1}}E_1^{b_1}...E_{n-1}^{b_{n-1}}\quad .\end{aligned}\tag{4.15}$$

The analogous vector $b_{\beta\alpha}^a(\beta)$ can be calculated in the reference frame of β. The two vectors are related by the formula

$$b_{\alpha\beta}^a(\alpha) = U_b^a(\alpha,\beta)b_{\beta\alpha}^b(\beta)\tag{4.16}$$

By construction these vectors are orthogonal to the face $f_{\alpha\beta}$ so that, from a mathematical point of view, they live on the dual Voronoï edge (α,β) (see appendix C). We choose the order of the the edges vectors E_i^a in formula 4.15 in such a

way that $b^a_{\alpha\beta}(\alpha)$ points toward the outside of the simplex α. Among the vectors $b^a_{\alpha\beta}(\alpha)$ of the same n-dimensional simplex there is the following identity

$$\sum_{\beta=1}^{n+1} b^a_{\alpha\beta}(\alpha) = 0 \quad . \tag{4.17}$$

The antisymmetric tensor $\mathcal{V}^{(h)}{}_{a_1 a_2}(\alpha)$ can be written as a bivector of the $b^a_{\alpha\beta}(\alpha)$

$$\mathcal{V}^{(h)\ ab}(\alpha) = \frac{1}{n!(n-2)!V(\alpha)} \left(b^a_{\alpha\alpha+1}(\alpha) b^b_{\alpha-1\alpha}(\alpha) - b^b_{\alpha\alpha+1}(\alpha) b^b_{\alpha-1\alpha}(\alpha) \right) \quad , \tag{4.18}$$

where $\alpha - 1$ and $\alpha + 1$ indicate the two n-simplices which have a face and the hinge h in common with α. The volume $V(\alpha)$ of the simplex α can be written as function of the $b_{\alpha\beta}$ in the following way

$$(V(\alpha))^{n-1} = \frac{1}{n!^n} \epsilon_{a_1 \dots a_n} \epsilon^{j_1 \dots j_n j} b^{a_1}_{j_1}(\beta) \dots b^{a_n}_{j_n}(\beta) \quad , \tag{4.19}$$

where the indices $(j_1, \dots, j_n)$ run over all possible values of the dual Voronoï edges (α, β) , $\beta = 1, \dots, n+1$ and j is a fixed index. The equation 4.19 does not depend from the choice of the index j in the ϵ-indices, or, in other words, it does not depend from the b that is left out.

As said in the previous section we can put the origin of the reference frame of α in the circumcenter. If $z^a_i(\alpha)$, $a = 1, \dots, n$, $i = 1, \dots, n+1$ are the circumcentric coordinates [83] of the $n+1$ vertices of α we have

$$\sum_{i=1}^{n+1} z^a_i(\alpha) = 0 \quad . \tag{4.20}$$

In these coordinates we can express, as it is easy to see, the vector $b^a_i(\alpha)$, $i = 1, \dots, n$ in the following way

$$b^a_i(\alpha) = \frac{1}{n!} \sum_{k \neq i} \epsilon^a_{a_1, \dots, a_n} \epsilon_i^{k j_1 \dots j_n} z^{a_1}_{j_1}(\alpha) \dots z^{a_{n-1}}_{j_{n-1}}(\alpha) \quad . \tag{4.21}$$

This equation can be inverted, so we have

$$z^a_i(\alpha) = \frac{1}{(n+1)!(n!V(\alpha))^{n-2}} \sum_{k \neq i} \epsilon^a_{a_1 \dots a_{n-1}} \epsilon_{i\,k}^{j_1 \dots j_{n-1}} b^{a_1}_{j_1}(\alpha) \dots b^{a_{n-1}}_{j_{n-1}}(\alpha) \quad . \tag{4.22}$$

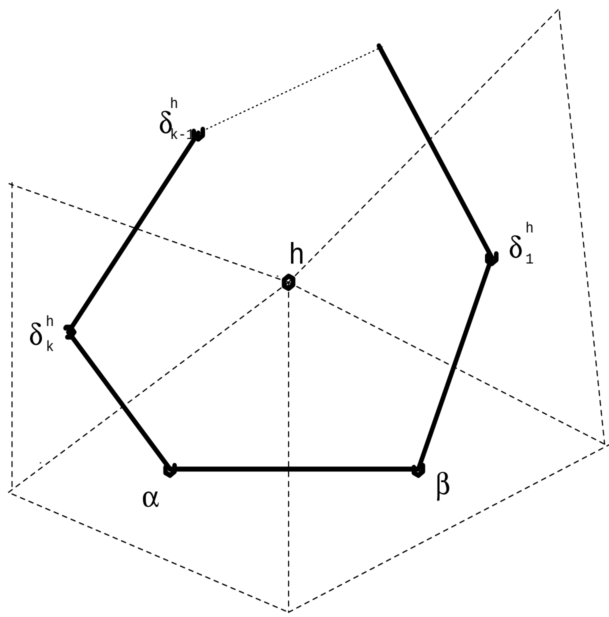

Figure 4.1: Dual Voronoï Plaquette

These two equations 4.21 4.22 show that there is a one to one correspondence between the circumcentric coordinates of the vertices of a simplex α and the $b_i(\alpha)$.

The gravitational action 4.8 can be written, by the equation 4.18, as a function of the $b^a_i(\alpha)$. Anyway we are still dealing with a second order formalism, since the connection matrices $U^a_b(\alpha,\beta)$ and the $b_{\alpha\beta}(\alpha)$ are both functions of the coordinates of the edge of the simplicial complex. Now we propose a first order formalism in which $U^a_b(\alpha,\beta)$ and $b_{\alpha\beta}(\alpha)$ are independent variables defined on the Voronoï edge (α,β). We consider equation 4.16 as a constraint of the theory. This constraint fix only n degree of freedom for each Voronoï edge (α,β), and not the $\frac{n(n-1)}{2}$ conditions that are necessary to determine $U^a_b(\alpha\beta)$. Pairwise 4.17 is the second constraint of the theory for each Voronoï vertex.

In the first order formalism we consider connection matrices more general than the Levi-Civita matrices defined by equation 4.3 .

This implies that in these cases the gravitational action in the form (4.8) could be dependent from the reference frame in which it is written since it is no longer true an equality like 4.13 . So if we define the following antisymmetric tensor on the plaquette h as

$$
\begin{aligned}
W^{(h)}_{c_1c_2}(\alpha) \quad &\equiv \quad \frac{1}{k_h+2}\Big(\mathcal{V}^{(h)}(\alpha) + U_{\alpha\beta}\mathcal{V}^{(h)}(\beta)U_{\beta\alpha} + ... \\
&+ \quad U_{\alpha\beta}...U_{\delta^h_{k-1}\delta^h_k}\mathcal{V}^{(h)}(\delta^h_k)U_{\delta^h_k\delta^h_{k-1}}...U_{\beta\alpha}\Big)_{c_1c_2} \quad .
\end{aligned} \tag{4.23}
$$

the action can be written in the form

$$
S \equiv -\frac{1}{2}\sum_h \mathrm{Tr}\left(U^h_{\alpha\alpha}W^h(\alpha)\right) \tag{4.24}
$$

The above action coincide with the previous action in second order formalism and

since (in matrix notation)

$$
W^{(h)}(\alpha) = U_{\alpha\beta}...U_{\delta_{i-1}\delta_i}W^{(h)}(\delta_i)U_{\delta_i\delta_{i-1}}...U_{\beta\alpha} \tag{4.25}
$$

the action (4.24) result to be independent form the frame in which it is written essentially for the same reasonings of the lemma (4.11).
Moreover the action (4.24) is invariant under the following set of transformations

$$
\begin{aligned}
U_{\alpha\beta} \quad &\mapsto \quad U'_{\alpha\beta} = O(\alpha)U_{\alpha\beta}O^{-1}(\beta) \\
b_{\alpha\beta}(\alpha) \quad &\mapsto \quad b'_{\alpha\beta}(\alpha) = O(\alpha)b_{\alpha\beta}(\alpha)
\end{aligned} \tag{4.26}
$$

where $O(\alpha)$ and $O(\beta)$ are two element of $SO(n)$.

We want to close this section by adding some consideration on the nature of the $b_{\alpha\beta}$. In the interior of every simplex σ, as we have remarked above, the space is flat. So in every point of the cotangent space to σ the metric is flat

$$
\mathbf{g}(\sigma) = \delta_{ab}\mathbf{e}^a(\sigma) \otimes \mathbf{e}^b(\sigma) \tag{4.27}
$$

where $\mathbf{e}^a(\sigma), a = 1, ..., n$ form an orthonormal dual base in the cotangent space of σ. Of course it is evident that in every point of the cotangent space of σ we can choose the same base $\mathbf{e}^a(\sigma)$. In the form (4.27) the metric coefficients are automatically diagonal and so we can say that trivially a n-dimensional simplex is a local inertial reference frame.
It seems more natural to choose as base the edges of the simplex σ because this makes contact with the geometry of the simplex σ. Let

$$
\mathbf{E}_\mu(\sigma) \ \mu = 1, ..., n \tag{4.28}
$$

n linear independent edge vectors. The other edge vectors can be expressed as linear combination of the n vectors above. Associated to these n-vectors we can fix (uniquely) n covectors of the dual base such that, by the definition (4.15),

$$\widehat{\mathbf{b}}^{\mu}(\sigma)(\mathbf{E}_{\nu}(\sigma)) = \delta^{\mu}_{\nu} \quad . \tag{4.29}$$

where

$$\widehat{\mathbf{b}}^{\mu}(\sigma) \equiv \frac{\mathbf{b}^{\mu}(\sigma)}{n!V(\sigma)}. \tag{4.30}$$

The transformation law between the couple of the orthonormal base $\mathbf{e}_a(\sigma)$ and their respective duals $\mathbf{e}^a(\sigma)$ and the couple of $\mathbf{E}_{\nu}(\sigma)$ and $\widehat{\mathbf{b}}^{\mu}(\sigma)$ are given once we fix the component of $\mathbf{E}_{\nu}(\sigma)$ and $\widehat{\mathbf{b}}^{\mu}(\sigma)$ in the base $\mathbf{e}_a(\sigma)$ and $\mathbf{e}^a(\sigma)$, more precisely

$$\begin{aligned} \mathbf{e}^a(\sigma) &= E^a_{\mu}(\sigma)\widehat{\mathbf{b}}^{\mu}(\sigma) \\ \mathbf{e}_a(\sigma) &= \widehat{b}^{\mu}_a(\sigma)\mathbf{E}_{\mu}(\sigma) \quad . \end{aligned} \tag{4.31}$$

In these basis we can write the metric tensor as

$$\begin{aligned} \mathbf{g}(\sigma) &= \delta^{ab}\widehat{b}^{\mu}_a(\sigma)\widehat{b}^{\nu}_b(\sigma)\mathbf{E}_{\mu}(\sigma) \otimes \mathbf{E}_{\nu}(\sigma) \\ \mathbf{g}(\sigma) &= \delta_{ab}E^a_{\mu}(\sigma)E^b_{\nu}(\sigma)\widehat{\mathbf{b}}^{\mu}(\sigma) \otimes \widehat{\mathbf{b}}^{\nu}(\sigma) \end{aligned} \tag{4.32}$$

It is now clear that we can do the following identification respect to the continuum theory

$$\begin{aligned} e^a_{\mu}(x) &\longmapsto E^a_{\mu}(\sigma) \\ e^{\mu}_a(x) &\longmapsto b^{\mu}_a(\sigma) \quad , \end{aligned} \tag{4.33}$$

where with $e^a_{\mu}(x)$ and $e^{\mu}_a(x)$ we indicate the n-bein of the continuum theory. The discrete n-bein, in analogy to the continuum case, are determined modulo the action of the orthogonal group because if we perform an orthogonal transformation on $E^a_{\mu}(\sigma)$ and $b^{\mu}_a(\sigma)$, the equations (4.32) remain form invariant.
Moreover the second equation of (4.32) says that the coefficient of the metric tensor in the base of the $\widehat{\mathbf{b}}^{\mu}(\sigma)$ are the usual metric coefficients written in Regge calculus

$$\begin{aligned} g_{\mu\mu}(\sigma) &= l^2_{\mu}(\sigma) \\ l^2_{\mu\nu}(\sigma) &= \frac{1}{2}(g_{\mu\mu}(\sigma) + g_{\nu\nu}(\sigma) - g_{\mu\nu}(\sigma)) \end{aligned} \tag{4.34}$$

where, in the standard notation, $l_\mu^2 \equiv |\mathbf{E}_\mu|^2$ and with $\mathbf{E}_{\mu\nu} = \mathbf{E}_\mu - \mathbf{E}_\nu$, $l_{\mu\nu}^2 = |\mathbf{E}_{\mu\nu}|^2$.

4.4 Remark on the Orientation

In the definitions (4.18), (4.23), (4.5) we have tacitly assumed that the boundary of the plaquettes is oriented. As a consequence, for example, we can establish that the Voronoï-edge (α, β), in the direction from α to β, is positive oriented, while the opposite Voronoï-edge $\beta\alpha$ is negative oriented. So this naturally fix a rule to define the bivectors $\mathcal{V}^{(h)\ c_1 c_2}(\alpha)$, $W^h(\alpha)$ and the holonomy matrix $U^h_{\alpha\alpha}$ along the positive direction. In particular we use the notation $\mathcal{V}^{(h)\ c_1 c_2}(\alpha) \equiv \mathcal{V}^{(h)\ c_1 c_2}_{\ \alpha\beta}(\alpha)$ to stress that in the definition 4.18 the positive direction is from α to β in order that in the bivector we have to use first the vector $b_{\alpha\beta}(\alpha)$ and after $b_{\alpha\delta_k}(\alpha)$. Pairwise we indicate as W^{+h} the antisymmetric tensor 4.23 that in these new notations is

$$\begin{aligned} W^{+(h)}_{\ c_1 c_2}(\alpha) &\equiv \frac{1}{k_h+2}\Big(\mathcal{V}^{(h)}_{\alpha\beta}(\alpha) + U_{\alpha\beta}\mathcal{V}^{(h)}_{\beta\delta_1}(\beta)U_{\beta\alpha} + \ldots \\ &+ U_{\alpha\beta}\ldots U_{\delta^h_{k-1}\delta^h_k}\mathcal{V}^{(h)}_{\delta^h_k\alpha}(\delta^h_k)U_{\delta^h_k\delta^h_{k-1}}\ldots U_{\beta\alpha}\Big)_{c_1 c_2} \quad , \end{aligned} \tag{4.35}$$

and $U^{+h}_{\ \alpha\alpha} \equiv U^h_{\alpha\alpha}$, that is the holonomy matrix in 4.5 is defined along the positive direction (α, β).

The negative direction is from α to δ^h_k, and as in the definitions above we have

$$\begin{aligned} W^{-(h)}_{\ c_1 c_2}(\alpha) &\equiv \frac{1}{k_h+2}\Big(\mathcal{V}^{(h)}_{\alpha\delta^h_k}(\alpha) + U_{\alpha\delta^h_k}\mathcal{V}^{(h)}_{\alpha\delta^h_k}(\delta^h_k)U_{\delta^h_k\alpha} + \ldots \\ &+ U_{\alpha\delta^h_k}\ldots U_{\delta^h_1\beta}\mathcal{V}^{(h)}_{\beta\alpha}(\beta)U_{\beta\delta^h_1}\ldots U_{\delta^h_k\alpha}\Big)_{c_1 c_2} \quad , \end{aligned} \tag{4.36}$$

and

$$U^{-h}_{\ \alpha\alpha} \equiv U_{\alpha\delta^h_k}U_{\delta^h_k\delta^h_{k-1}}\ldots U_{\beta\alpha} \quad . \tag{4.37}$$

Now we are going to prove the following lemma:

Lemma 4.4.1 *The action 4.24 is independent from the chosen orientation for the boundary of the plaquettes* h *in the sense that*

$$S \equiv -\frac{1}{2}\sum_h \mathrm{Tr}\left(U^{+h}_{\ \alpha\alpha}W^{+h}(\alpha)\right) = -\frac{1}{2}\sum_h \mathrm{Tr}\left(U^{-h}_{\ \alpha\alpha}W^{-h}(\alpha)\right) \quad . \tag{4.38}$$

Dim: since the trace of a matrix and its transposed are the same we can write

$$-\frac{1}{2}\mathrm{Tr}\left(U^{+h}_{\alpha\alpha}W^{+h}(\alpha)\right) = -\frac{1}{2}\mathrm{Tr}\left(U^{+h}_{\alpha\alpha}W^{+h}(\alpha)\right)^{T} = -\frac{1}{2}\mathrm{Tr}\left({U^{+h}_{\alpha\alpha}}^{T}W^{+h}(\alpha)^{T}\right) \quad . \tag{4.39}$$

Now we have that

$${U^{+h}_{\alpha\alpha}}^{T} = U_{\alpha\delta^h_k}U_{\delta^h_k\delta^h_{k-1}}...U_{\beta\alpha} \tag{4.40}$$

and

$$\begin{aligned} W^{+h}(\alpha)^{T} &= \frac{1}{k_h+2}\Big(\mathcal{V}^{(h)}_{\alpha\delta^h_k}(\alpha) + U_{\alpha\beta}\mathcal{V}^{(h)}_{\beta\alpha}(\beta)U_{\beta\alpha} + ... \\ &+ U_{\alpha\beta}...U_{\delta^h_{k-1}\delta^h_k}\mathcal{V}^{(h)}_{\delta^h_k\delta^h_{k-1}}(\delta^h_k)U_{\delta^h_k\delta^h_{k-1}}...U_{\beta\alpha}\Big) \quad . \end{aligned} \tag{4.41}$$

Consider the orthogonal matrix

$$\Gamma = U_{\alpha\beta}U_{\beta\delta^h_1}...U_{\delta^h_{k-1}\delta^h_k} \tag{4.42}$$

and manipulate the last trace in 4.39 in the following way

$$-\frac{1}{2}\mathrm{Tr}\left({U^{+h}_{\alpha\alpha}}^{T}W^{+h}(\alpha)^{T}\right) = -\frac{1}{2}\mathrm{Tr}\left(\Gamma^{T}{U^{+h}_{\alpha\alpha}}^{T}\Gamma\Gamma^{T}W^{+h}(\alpha)^{T}\Gamma\right) \quad . \tag{4.43}$$

A straightforward calculation shows that

$$\begin{aligned} \Gamma^{T}W^{+h}(\alpha)^{T}\Gamma &= \frac{1}{k_h+2}\Big(\mathcal{V}^{(h)}_{\delta^h_k\delta^h_{k-1}}(\delta^h_k) + U_{\delta^h_k\delta^h_{k-1}}\mathcal{V}^{(h)}_{\delta^h_{k-1}\delta^h_{k-2}}(\delta^h_{k-1})U_{\delta^h_{k-1}\delta^h_k} + ... \\ &+ U_{\delta^h_k\delta^h_{k-1}}...U_{\beta\alpha}\mathcal{V}^{(h)}_{\alpha\delta^h_k}(\alpha)U_{\alpha\beta}...U_{\delta^h_{k-1}\alpha^h_k}\Big) \quad , \end{aligned} \tag{4.44}$$

and

$$\Gamma^{T}{U^{+h}_{\alpha\alpha}}^{T}\Gamma = U_{\delta^h_k\delta^h_{k-1}}U_{\delta^h_{k-1}\delta^h_{k-2}}...U_{\alpha\delta^h_k} \quad . \tag{4.45}$$

These last two equations imply that

$$-\frac{1}{2}\mathrm{Tr}\left({U^{+h}_{\alpha\alpha}}^{T}W^{+h}(\alpha)^{T}\right) = -\frac{1}{2}\mathrm{Tr}\left(U^{-h}_{\delta^h_k\delta^h_k}W^{-h}(\delta^h_k)\right) \tag{4.46}$$

Since the trace 4.46 is independent from the reference frame where it is written, we have the equality 4.38.

This property of the action is necessary not to have ambiguity in defining it. As we have seen on each hinge there is no preferred orientation (c.f. section 4.2) and, as is easy to see, no orientation on the corresponding Voronoï-dual plaquettes even if the manifold is oriented.

4.5 First Order Field Equations for Small Deficit Angles

As remarked in [72], in the second order formalism the action (4.8) is equivalent to the Regge action for small deficit angles θ_h (since in this case $sin\theta_h \approx \theta_h$). In the first order formalism we don't have angles θ_h, but the only variables related to the deficit angles are the connection matrices $U_{\alpha\beta}$. Then we assume, by definition, that the *small deficit angles approximation* in the first order formalism is the passage from the group variables $U_{\alpha\beta}$ of SO(n) to the algebra variables $\phi_{\alpha\beta}$ of so(n). As a consequence the connection matrices can be written in the form

$$U_{\alpha\beta} = I + \epsilon\ \phi_{\alpha\beta} + o(\epsilon) \tag{4.47}$$

This implies, as it is easy to verify, that the only gauge transformations (4.26) which can be compatible with the equation (4.47) are of the type $O(\alpha) = I$, or, in other words, the approximation in the form (4.47) is also a gauge fixing.
In order to avoid technical complication, that we shall discuss in the next section, we now suppose to substitute the constraint $b_{\alpha\beta}(\alpha) = U_{\alpha\beta} b_{\beta\alpha}(\beta)$ in the action for each Voronoï edge (α, β).

A straightforward expansion of the action up to the first order shows that

$$S = -\frac{1}{2}\epsilon \sum_h \mathrm{Tr}\left((\phi_{\alpha\beta} + \phi_{\beta\delta_1^h} \ldots + \phi_{\delta_k^h\alpha})\ {}^0W^h(\alpha)\right) + o(\epsilon) \tag{4.48}$$

where ${}^0W^{h\ c_1c_2}$ is the bivector $W^{h\ c_1c_2}$ to the zero order in which we have done for each matrix the approximation (4.47)and for each Voronoï-edge we have solved the constraint (4.15) up to the first order, that is to say

$$b_{\alpha\beta}(\alpha) = (I + \epsilon\ \phi_{\alpha\beta}) b_{\beta\alpha}(\beta) + o(\epsilon) \tag{4.49}$$

Demanding that the action be stationary under the variation respect to $(\phi_{\alpha\beta})$ we obtain

$$\frac{\delta S}{\delta \phi_{\alpha\beta\ c_1 c_2}} = \epsilon \sum_{h \in (\alpha\beta)} {}^0W^{h\ c_1 c_2} + o(\epsilon) = 0 \tag{4.50}$$

where $h \in (\alpha\beta)$ means the sum over all plaquettes h to which the Voronoï-edge $\alpha\beta$ belongs. The tensor ${}^0W^{h\ c_1 c_2}$ is an antisymmetric tensor of type $(2,0)$ in n-dimensions, so has $\frac{n(n-1)}{2}$ independent components, the same independent components of $\phi_{\alpha\beta}$ that have to be determined.

Let's consider the face between the simplices α and β, dual to the Voronoï-edge $\alpha\beta$. The equations ([72]) that determine the Levi-Civita connection $U_{\alpha\beta}$ between the reference frames of the two simplices , to the first order (4.47) are:

$$\begin{aligned} (z_1^a(\alpha) - z_n^a(\alpha)) &= (I + \epsilon\ \phi_{\alpha\beta})^a_b (z_1^b(\beta) - z_n^b(\beta)) \\ \cdots\cdots\cdots\cdots & \cdots\ \cdots\cdots\cdots\cdots \\ (z_{n-1}^a(\alpha) - z_n^a(\alpha)) &= (I + \epsilon\ \phi_{\alpha\beta})^a_b (z_{n-1}^b(\beta) - z_n^b(\beta)) \quad . \end{aligned} \tag{4.51}$$

The $z_i^a(\alpha)$, $a = 1, ..., n$ are the coordinates of the vertices $i = 1, ..., n$ of the face in the reference frame of the circumcenter in α. The circumcentric coordinates 4.20 of $n+1$ vertices of a n-dimensional simplex are in one to one correspondence with the $n+1$ $b_i^a(\alpha)$, $\{i = 1, ..., n+1\}$. Equations (4.51) can be seen as an equation for determining $\phi_{\alpha\beta}$ as function of the $z_i^a(\alpha)$ (and then of the $b_i^a(\alpha)$). Each $\phi_{\alpha\beta}$, being an antisymmetric matrix in n-dimension, has $\frac{n(n-1)}{2}$ degree of freedom. The number of independent equations are $n(n-1)$. Anyway there is the following identity among the edge components of the $n-1$-dimensional face (α, β)

$$\sum_{h \in (\alpha\beta)} \mathcal{V}^{h\ c_1 c_2}(\alpha) = 0 \quad , \tag{4.52}$$

The (4.52) fixes $\frac{n(n-1)}{2}$ degrees of freedom, so that the (4.51) together (4.52) has $\frac{n(n-1)}{2}$ linear independent equations. By linearity there is for $\phi_{\alpha\beta}$ only one solution. Anyway this is the connection that we adopt in the second order formalism. We want to see if this connection, that is the Levi-Civita or Regge connection, satisfies equation (4.50). If the connection is Regge we have (in matrix notation)

$$\mathcal{V}^{(h)}(\alpha) = U_{\alpha\beta}\mathcal{V}^{(h)}(\beta)U_{\beta\alpha} = ... = U_{\alpha\beta}...U_{\delta^h_{k-1}\delta^h_k}\mathcal{V}^{(h)}(\delta^h_k)U_{\delta^h_k\delta^h_{k-1}}...U_{\beta\alpha} \tag{4.53}$$

so that to the zero order

$$\mathcal{V}^{(h)}(\alpha) = \mathcal{V}^{(h)}(\beta) + O(\epsilon) = \ldots = \mathcal{V}^{(h)}(\delta_k^h) + O(\epsilon) \quad . \tag{4.54}$$

These facts imply that

$$\epsilon \sum_{h\in(\alpha\beta)} {}^0W^{h\ c_1c_2} = \sum_{h\in(\alpha\beta)} \left(\epsilon\mathcal{V}^{h\ c_1c_2}(\alpha) + \epsilon O(\epsilon)\right) = 0 + O(\epsilon^2) \tag{4.55}$$

that is to say that Regge connection is solution of our first order equations for the connection matrices in the limit of small deficit angles.

4.6 First Order Field Equations: the General Case

In the previous section we have seen that in the case of small deficit angles the Regge calculus is solution of the first order field equations.

We are now going to deal with the general problem. We want to derive the equation of motion by varying the action respect to $U_{\alpha\beta}$ and $b_{\alpha\beta}$. This formulation of gravity on lattice is close to the popular Ashtekar's variables [86] in non perturbative canonical continuum quantum gravity (see also [87] for a covariant version). Of course we have to take in account the constraints (4.16) and (4.17), so in order to perform independent variation of $U_{\alpha\beta}$ and $b_{\alpha\beta}$ it is necessary to introduce the constraints in the action by using Lagrange multipliers. Then the first order action will be

$$\begin{aligned} S \ &\equiv\ -\frac{1}{2}\sum_h \mathrm{Tr}\left(U^h_{\alpha\alpha}W^h(\alpha)\right) + \sum_{(\alpha\beta)} \lambda_{\alpha\beta}\left(b_{\alpha\beta}(\alpha) - U_{\alpha\beta}b_{\beta\alpha}(\beta)\right) \\ &+\ \sum_{(\alpha\beta)} \mathrm{Tr}\left(\tilde{\lambda}^{(\alpha\beta)}(U_{\alpha\beta}U^T_{\alpha\beta} - I)\right) + \sum_\alpha \mu(\alpha)\left(\sum_{\beta=1}^{n+1} b_{\alpha\beta}(\alpha)\right) \end{aligned} \tag{4.56}$$

where $\lambda^{(\alpha\beta)}$ and $\mu(\alpha)$ are n-dimensional vectors and are Lagrange multipliers. $\tilde{\lambda}^{(\alpha\beta)}$ is an $n\times n$ matrix. The constraint $U_{\alpha\beta}U^T_{\alpha\beta} - I$ is introduced to restrict the variation of $U_{\alpha\beta}$ on the group $SO(n)$.

Let's introduce the following lemma:

Lemma: *The action, in matrix notation,*

$$S' \equiv \mathrm{Tr}(\Lambda A) + \mathrm{Tr}\left(\lambda(\Lambda\Lambda^T - I)\right) \tag{4.57}$$

gives the following equation of motion if we assume that the variation respect to Λ be stationary

$$(\Lambda A) = (\Lambda A)^T \quad . \tag{4.58}$$

Dim: If we consider the variation of the action respect to Λ and Λ^T, using the property that $\mathrm{Tr}(M) = \mathrm{Tr}(M^T)$, we have

$$\delta S' = \frac{1}{2}\mathrm{Tr}(\delta\Lambda A + A^T\delta\Lambda^T) + \frac{1}{2}\mathrm{Tr}\lambda(\delta\Lambda\Lambda^T + \Lambda\delta\Lambda^T) + \frac{1}{2}\mathrm{Tr}(\delta\Lambda\Lambda^T + \Lambda\delta\Lambda^T)\lambda^T \tag{4.59}$$

that implies

$$\begin{aligned} \frac{\delta S'}{(\delta\Lambda)\Lambda^T} &= \Lambda A + \lambda + \lambda^T = 0 \\ \frac{\delta S'}{\Lambda(\delta\Lambda^T)} &= (\Lambda A)^T + \lambda^T + \lambda = 0 \quad , \end{aligned} \tag{4.60}$$

from which, subtracting term by term these two equations we have (4.58).
We can apply the lemma to the action (4.56) for the variation respect to $U_{\alpha\beta}$, we obtain the following field equations

$$\sum_{h\in(\alpha\beta)} (U^h_{\alpha\alpha} W^h(\alpha))_{ij} - \lambda_{\alpha\beta\ i} b_{\alpha\beta}(\alpha)_{\ j} = \sum_{h\in(\alpha\beta)} (U^h_{\alpha\alpha} W^h(\alpha))^T_{ij} - \lambda_{\alpha\beta\ j} b_{\alpha\beta}(\alpha)_{\ i} \tag{4.61}$$

The next step will consist of determining the field equations for the variations of $b_{\alpha\beta}$. For this aim it is necessary to determine the quantity $\frac{\partial V(\alpha)}{\partial b^a_{\alpha\beta}}$. We recall that in the circumcentric coordinates, the coordinates of the i-th vertex $z^a_i(\beta)$ is given by formula 4.68

We can write the formula 4.19 in a way not dependent from the chosen index j. If we call $(V^{(j)}(\alpha))^{n-1}$ equation 4.19, we have that it can be rewritten as

$$(V^*(\alpha))^{n-1} = \sqrt[n+1]{((V^{(1)}(\alpha))^{n-1}) \dots ((V^{(n+1)}(\alpha))^{n-1})} \quad , \tag{4.62}$$

in such a way that it does not depend from any index. Evaluating the derivative of this last expression we get,

$$\frac{\partial\left(\frac{1}{V(\alpha))}\right)}{\partial b^a_i(\alpha)} = -\frac{1}{n!(n-1)V^2(\alpha)} z^i_a(\alpha) \quad . \tag{4.63}$$

We are now ready to derive the field equations for the variations of the $b_{\alpha\beta}$. The starting action is (4.56). Furthermore we have remarked that the action is invariant under the orientation of the boundary of each plaquette, so we can write it in an orientation independent way

$$\begin{aligned} S &\equiv -\frac{1}{4}\sum_h \mathrm{Tr}\left(U^h_{\alpha\alpha}W^h(\alpha)+U^{T\ h}_{\ \alpha\alpha}W^{T\ h}(\alpha)\right)+\sum_{(\alpha\beta)}\lambda^{(\alpha\beta)}\left(b_{\alpha\beta}(\alpha)-U_{\alpha\beta}b_{\beta\alpha}(\beta)\right) \\ &+ \sum_\alpha \mu(\alpha)\left(\sum_{\beta=1}^{n+1} b_{\alpha\beta}(\alpha)\right) \quad . \end{aligned} \tag{4.64}$$

A straightforward calculation show that:
with the notation

$$U^{h\ \ ij}_{\alpha\alpha} - U^{h\ \ ji}_{\alpha\alpha} \equiv \Omega^{h\ \ ij}_{\alpha\alpha} \tag{4.65}$$

$$\begin{aligned} \frac{\partial S}{\partial b^i_{\alpha\beta}(\alpha)} &\equiv \sum_{h\in(\alpha\beta)} \frac{1}{4V(\alpha)}\left(\Omega^h_{\alpha\alpha\ ij}b^j_{\alpha\delta^h_k} - \frac{1}{n!(n-1)}\mathrm{Tr}(\Omega^h_{\alpha\alpha}\mathcal{V}^h(\alpha))z^{\alpha\beta}_i(\alpha)\right) \\ &+ \lambda_{i\ \alpha\beta} - \mu_i(\alpha) = 0 \quad . \end{aligned} \tag{4.66}$$

In the field equations (4.66) and (4.61) we have to determine also the Lagrange multipliers $\lambda^i_{\alpha\beta}$ and $\mu^i(\alpha)$. We now propose a method for determining them. In each simplex α, or, equivalently, in each point of the dual complex, we have $n+1$ vectors $b^i_{\alpha\beta}(\alpha), b^i_{\alpha\delta^1}(\alpha), \ldots, b^i_{\alpha\delta^n}(\alpha)$ and among them there is the constraint (4.17). Let us choose n of this vectors for example $b^i_{\alpha\beta}(\alpha), \ldots, b^i_{\alpha\delta_{n-1}}(\alpha)$, in other words we are solving the constraint (4.17). This is a base in the tangent space of the n-simplex α so that we can write each Lagrange multiplier, such that $\lambda^h_{\alpha\beta}$, considered as a vector in tangent space of α as

$$\lambda^h_{\alpha\beta} = c^1_{\alpha\beta}b^h_{\alpha\beta}(\alpha)\ldots c^n_{\alpha\beta}b^h_{\alpha\delta_{n-1}}(\alpha) \tag{4.67}$$

The set of n vectors $b^i_{\alpha\beta}(\alpha), \ldots, b^i_{\alpha\delta_{n-1}}(\alpha)$ are in correspondence with the circumcentric coordinate

$$z^a_{n+1}(\alpha) = \frac{1}{(n+1)!(n!V(\alpha))^{n-2}}\sum_{k\neq n+1}\epsilon^{\ a}_{a_1\ldots a_{n-1}}\epsilon^{j_1\ldots j_{n-1}}_{n+1\ k}b^{a_1}_{j_1}(\alpha)\ldots b^{a_{n-1}}_{j_{n-1}}(\alpha) \tag{4.68}$$

where the indices $(j_1, ..., j_{n-1}, k)$ run over $\alpha\beta, \alpha\delta_1, ..., \alpha\delta_{n-1}$ and in this case $n+1 = \alpha\delta_n$.
From a geometrical point of view elementary arguments show that $z^a_{n+1}(\alpha)$ is the circumcentric coordinate of the vertex of α which is the intersection of the n $n-1$-dimensional faces whose normals are $b^i_{\alpha\beta}(\alpha), ..., b^i_{\alpha\delta_{n-1}}(\alpha)$.
Pairwise to each $b_{\alpha\delta_i}(\alpha)$ we can associate a circumcentric coordinate $z^a_{\alpha\delta_i}(\alpha)$ obtained as in the definition 4.68 in which $n+1 \mapsto i = \alpha\delta_i$ and $(j_1, ..., j_{n-1}, k)$ now run over $\alpha\beta, \alpha\delta_1, ..., \alpha\delta_n$ in which, of course, $\alpha\delta_i$ is omitted. In this way we can make a one to one correspondence within each $b_{\alpha\delta_i}(\alpha)$ and the edge of the simplex α $E_{\alpha\delta_i}(\alpha)$ defined in the circumcentric coordinate

$$E^a_{\alpha\delta_i}(\alpha) \equiv z^a_{\alpha\delta_i}(\alpha) - z^a_{n+1}(\alpha) \tag{4.69}$$

It is straightforward to see that

$$(b_{\alpha\delta_i}(\alpha))^a \left(E_{\alpha\delta_j}(\alpha)\right)_a = n!V(\alpha)\delta_{ij} \quad . \tag{4.70}$$

Consider the equation (4.67), project it along the edge $E^h_{\alpha\beta}$ and repeat the procedure for each $E^h_{\alpha\delta_i}, i = 1, ..., n-1$. We will obtain

$$\begin{aligned} c^1_{\alpha\beta} &= \frac{1}{n!V(\alpha)}(\lambda^h_{\alpha\beta}E_{\alpha\beta\ h}) \\ c^{i+1}_{\alpha\beta} &= \frac{1}{n!V(\alpha)}(\lambda^h_{\alpha\beta}E_{\alpha\delta_i\ h})\ \ i = 1, ..., n-1 \end{aligned} \tag{4.71}$$

In other words for determining $\lambda_{\alpha\beta}$ we have to determine its projections along the above edges. For this purpose we project equation (4.61) along $E^i_{\alpha\delta_i}(\alpha)$ and $E^j_{\alpha\beta}(\alpha)$ and we find that

$$\lambda^h_{\alpha\beta}E_{\alpha\delta_i\ h}(\alpha) = -\frac{1}{V(\alpha)}\sum_{h\in(\alpha\beta)}\left((U^h_{\alpha\alpha}W^h(\alpha))_{pq} - (U^h_{\alpha\alpha}W^h(\alpha))_{qp}\right)E^p_{\alpha\delta_i}(\alpha)E^q_{\alpha\beta}(\alpha) \quad , \tag{4.72}$$

which fixes the coefficients $c^i_{\alpha\beta},\ \ i = 2, ..., n$. A problem arises with the coefficient $c^1_{\alpha\beta}$, in fact equation (4.61) does not determine the component of $\lambda^i_{\alpha\beta}$ in the direction of $b^i_{\alpha\beta}(\alpha)$, because the following transformation $\lambda^i_{\alpha\beta} \mapsto \lambda^i_{\alpha\beta} + \alpha b^i_{\alpha\beta}$ does not affect the equation. It seems necessary to use the field equation for $b_{\alpha\beta}$ for evaluating $c^1_{\alpha\beta}$.
Anyway if we project (4.61) only along $E^j_{\alpha\beta}(\alpha)$ we obtain that

$$\lambda^i_{\alpha\beta} = \frac{1}{V(\alpha)}\left(\lambda^h_{\alpha\beta}E_{\alpha\beta\ h}(\alpha)\right)b^i_{\alpha\beta}(\alpha) + \sum_{h\in(\alpha\beta)}\left((U^h_{\alpha\alpha}W^h(\alpha))_{ij} - (U^h_{\alpha\alpha}W^h(\alpha))_{ji}\right)E^j_{\alpha\beta}(\alpha) \tag{4.73}$$

which, substituted in the starting equation, gives the field equation for $U_{\alpha\beta}$ without Lagrange multipliers

$$\sum_{h\in(\alpha\beta)}\left((U^h_{\alpha\alpha}W^h(\alpha))_{ij} - (U^h_{\alpha\alpha}W^h(\alpha))_{ji}\right) = \sum_{h\in(\alpha\beta)}\left((U^h_{\alpha\alpha}W^h(\alpha))_{ik} - (U^h_{\alpha\alpha}W^h(\alpha))_{ki}\right)$$
$$\times E^k_{\alpha\beta}(\alpha)b^j_{\alpha\beta}(\alpha) - \sum_{h\in(\alpha\beta)}\left((U^h_{\alpha\alpha}W^h(\alpha))_{jk} - (U^h_{\alpha\alpha}W^h(\alpha))_{kj}\right)E^k_{\alpha\beta}(\alpha)b^i_{\alpha\beta}(\alpha) \tag{4.74}$$

Of course the reasoning for the Voronoï-edge $\alpha\beta$ can be applied to other Voronoi edges $\alpha\delta_i$. The unknown coefficients will be $c^{i+1}_{\alpha\delta_i}$, $i = 2, ..., n-1$.
In the reference frame α we write the equation for the variation respect to $b_{\alpha\beta}$ and $b_{\alpha\delta_i}$ in the following manner

$$\begin{aligned} B^a_{\alpha\beta} + \lambda^a_{\alpha\beta} + \mu^a(\alpha) &= 0 \\ B^a_{\alpha\delta_i} + \lambda^a_{\alpha\delta_i} + \mu^a(\alpha) &= 0 \end{aligned} \tag{4.75}$$

where in the symbols B we have summarized all the terms in the equations (4.64) which don't contain the Lagrange multipliers. We recall that the expansion of $\lambda_{\alpha\beta}$ in the base of b is given by (4.67), pairwise

$$\lambda^a_{\alpha\delta_i} = c^1_{\alpha\delta_i}b^a_{\alpha\beta} + ... + c^{i+1}_{\alpha\delta_i}b^a_{\alpha\delta_i} + ... + c^n_{\alpha\delta_{n-1}}b^a_{\alpha\delta_{n-1}} \tag{4.76}$$

Subtracting the two equations (4.75) between them, we have

$$(B^a_{\alpha\beta} - B^a_{\alpha\delta_i}) + (c^1_{\alpha\beta} - c^1_{\alpha\delta_i})b^a_{\alpha\beta} + ... + (c^n_{\alpha\beta} - c_{\alpha\delta_{n-1}})b^a_{\alpha\delta_{n-1}} = 0 \quad . \tag{4.77}$$

Projecting along $(E_{\alpha\beta}(\alpha))_a$ we get

$$c^1_{\alpha\beta} = c^1_{\alpha\delta_i} - \frac{1}{n!V(\alpha)}(B^a_{\alpha\beta} - B^a_{\alpha\delta_i})(E_{\alpha\beta}(\alpha))_a \tag{4.78}$$

in this way we determine $c^1_{\alpha\beta}$, and a similar procedure gives

$$c^{i+1}_{\alpha\delta_i} = c^{i+1}_{\alpha\beta} - \frac{1}{n!V(\alpha)}(B^a_{\alpha\delta_i} - B^a_{\alpha\beta})(E_{\alpha\delta_i}(\alpha))_a \tag{4.79}$$

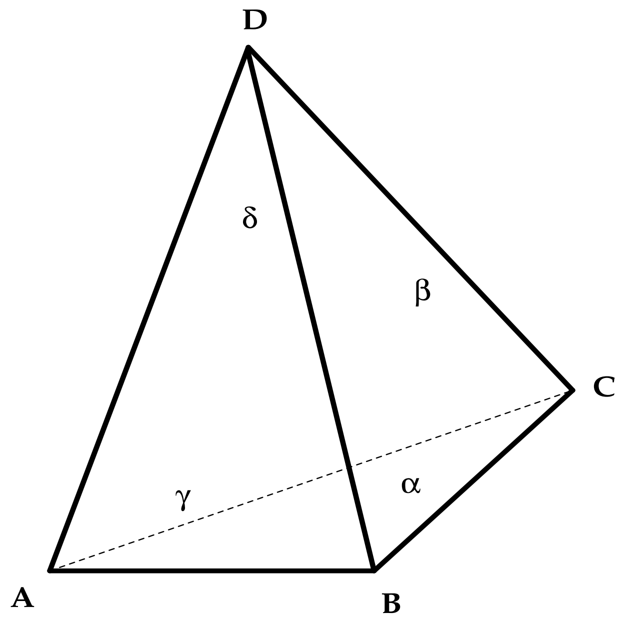

Figure 4.2: The Most Elementary Triangulation of the 2-Sfere

Once we have determined $\lambda^a_{\alpha\beta}$ and $\lambda^a_{\alpha\delta_i}$ we can determine $\mu^a(\alpha)$ by summing the two equations

$$\mu^a(\alpha) = -\frac{1}{2}(\lambda^a_{\alpha\beta} + \lambda^a_{\alpha\delta_i}) - (B^a_{\alpha\beta} + B^a_{\alpha\delta_i}) \tag{4.80}$$

We have used the equations for $b_{\alpha\beta}$ and $b_{\alpha\delta_i}$ to determine $\lambda^a_{\alpha\beta}$, $\lambda_{\alpha\delta_i}$, and μ^a_α. To determine $b_{\alpha\beta}$ and $b_{\alpha\delta_i}$ we can solve the equations for $b_{\beta\alpha}(\beta)$ and $b_{\delta_i\alpha}(\delta_i)$ in which the Lagrange multipliers $\mu^a(\beta)$, $\mu^a(\delta_i)$ have been determined by two couple of different equations. Then we will use the constraint $b_{\alpha\beta}(\alpha) = U_{\alpha\beta}b_{\beta\alpha}(\beta)$ and $b_{\alpha\delta_i}(\alpha) = U_{\alpha\delta_i}b_{\delta_i\alpha}(\delta_i)$ and so on. In this method, at the end, the only unknown quantity is the Lagrange multiplier $\lambda^a_{\alpha\delta_n}$ corresponding to the Voronoï-edge $\alpha\delta_n$ and to $b_{\alpha\delta_n}$. Anyway we have that $b_{\alpha\delta_n}$ is determined once we know all the other b and we can use the equation for $b_{\alpha\delta_n}$ to determine $\lambda_{\alpha\delta_n}$ as function of the other b.

Let do an example that show how this method work in practice. Consider the boundary of a tetrahedron as a two dimensional triangulation of the sphere

Let call α the triangle ABC, β the triangle BCD, γ the triangle ACD and δ the triangle ABD.

The edge BD is the face (one-dimensional) between α and β, $b_{\alpha\beta}(\alpha)$ will be the vector in α perpendicular to this face. DC is the face between β and γ, and $b_{\beta\gamma}(\beta)$

the vector perpendicular to it. AD is the face between γ and α, and $b_{\gamma\alpha}(\gamma)$ is the vector perpendicular to it, and so on. AC is perpendicular to the Voronoï-edge $\alpha\gamma$, BC to $(\beta\delta)$, AB to $\alpha\delta$.

Let's write the equation of motion for the variation of the b for this system:

$$
\begin{aligned}
B^a_{\alpha\beta} + \lambda^a_{\alpha\beta} + \mu^a(\alpha) &= 0 \quad \text{edge } (\alpha\beta)\\
B^a_{\alpha\gamma} + \lambda^a_{\alpha\gamma} + \mu^a(\alpha) &= 0 \quad \text{edge } (\alpha\gamma)\\
B^a_{\alpha\delta} + \lambda^a_{\alpha\delta} + \mu^a(\alpha) &= 0 \quad \text{edge } (\alpha\delta)\\
B^a_{\beta\alpha} - (\lambda_{\alpha\beta} U_{\alpha\beta})^a + \mu^a(\beta) &= 0 \quad \text{edge } (\beta\alpha)\\
B^a_{\beta\gamma} + \lambda^a_{\beta\gamma} + \mu^a(\beta) &= 0 \quad \text{edge } (\beta\gamma)\\
B^a_{\beta\delta} + \lambda^a_{\beta\delta} + \mu^a(\beta) &= 0 \quad \text{edge } (\beta\delta)\\
B^a_{\gamma\beta} - (\lambda_{\beta\gamma} U_{\beta\gamma})^a + \mu^a(\gamma) &= 0 \quad \text{edge } (\gamma\beta)\\
B^a_{\gamma\alpha} - (\lambda_{\alpha\gamma} U_{\alpha\gamma})^a + \mu^a(\gamma) &= 0 \quad \text{edge } (\gamma\alpha)\\
B^a_{\gamma\delta} + \lambda^a_{\gamma\delta} + \mu^a(\gamma) &= 0 \quad \text{edge } (\gamma\delta)\\
B^a_{\delta\alpha} - (\lambda_{\alpha\delta} U_{\alpha\delta})^a + \mu^a(\delta) &= 0 \quad \text{edge } (\delta\alpha)\\
B^a_{\delta\beta} - (\lambda_{\beta\delta} U_{\beta\delta})^a + \mu^a(\delta) &= 0 \quad \text{edge } (\delta\beta)\\
B^a_{\delta\gamma} - (\lambda_{\gamma\delta} U_{\gamma\delta})^a + \mu^a(\delta) &= 0 \quad \text{edge } (\delta\gamma)
\end{aligned}
\tag{4.81}
$$

From the first two equations we choose to determine $\lambda^a_{\alpha\beta}$, $\lambda^a_{\alpha\gamma}$ and $\mu^a(\alpha)$ by the above method. The equation for the Voronoï-edge $\alpha\delta$ will be used to determine $\lambda^a_{\alpha\delta}$. The equations for the Voronoï-edges $\beta\gamma$ and $\beta\delta$ will be used to determine $\lambda^a_{\beta\gamma}$, $\lambda_{\beta\delta}$ and $\mu^a(\beta)$. The equation for the Voronoï-edge $\beta\alpha$ will be used to determine $b_{\beta\alpha}$. From the constraint $b_{\alpha\beta} = U_{\alpha\beta} b_{\beta\alpha}$ we determine $b_{\alpha\beta}$ too. The equation for the Voronoï-edge $\gamma\beta$ may be used to determine $\mu^a(\gamma)$. The equation for $\gamma\alpha$ may be seen as an equation for determining $b_{\gamma\alpha}$, and so we determine $b_{\alpha\gamma}$ too. The equation for $\gamma\delta$ may be used to determine $\lambda_{\gamma\delta}$. The equation for $\delta\alpha$ determine $\mu^a(\delta)$, those for $\delta\beta$ and $\delta\gamma$ determine $b_{\delta\beta}$ and $b_{\delta\gamma}$ and so also $b_{\beta\delta}$ and $b_{\gamma\delta}$. The other b are determined from the constraint on the b (4.17).

In this way we determine all the Lagrange multiplier as functions of the $U_{\alpha\beta}$ and $b_{\alpha\beta}$. Using also the constraint we have that all the field equations fix a number of degrees of freedom that is equal to the number of degrees of freedom of $U_{\alpha\beta}$ and $b_{\alpha\beta}$.

4.7 Quantization of the Model

In this section we shall discuss the quantum measure to associate to the previous classical action. The action 4.24 is invariant under the action of the group $SO(n)$, the gauge group (4.26).
Let us fix the following notation: we define

$$\mu(b_{\alpha\beta}(\alpha)) \equiv db^1_{\alpha\beta}(\alpha)...db^n_{\alpha\beta}(\alpha) \tag{4.82}$$

and with

$$\mu(U_{\alpha\beta}) \tag{4.83}$$

the Haar measure of $SO(n)$. The partition function for this theory will be:

$$\begin{aligned} Z &= \int e^{\frac{1}{2}\sum_h \mathrm{Tr}\left(U^h_{\alpha\alpha} W^h(\alpha)\right)} \prod_\alpha \delta(\sum_{\beta=1}^{n+1} b_{\alpha\beta}(\alpha)) \\ &\quad \prod_{\alpha\beta} \delta\left(b_{\alpha\beta}(\alpha) - U_{\alpha\beta} b_{\beta\alpha}(\beta)\right) \mu(U_{\alpha\beta})\mu(b_{\alpha\beta}) \end{aligned} \tag{4.84}$$

in which the first product $\prod_\alpha$ is a product over all vertices of the dual complex, and the second product $\prod_{\alpha\beta}$ is the product over all the edges starting from the vertex α. The second delta function $\delta\left(b_{\alpha\beta}(\alpha) - U_{\alpha\beta} b_{\beta\alpha}(\beta)\right)$ has to be introduced one time for each edge, so that if we have introduced it for the edge $\alpha\beta$, we have not to introduce it for the edge $\beta\alpha$ in the reference frame β. It is straightforward to see that the measure of the partition function is invariant under the gauge transformation (4.26). In fact if we perform a gauge transformation of the type (4.26), the modified terms of the measure are:

$$\delta\left(\sum_{\beta=1}^{n+1} b'_{\alpha\beta}(\alpha)\right) \delta\left(b'_{\alpha\beta}(\alpha) - U'_{\alpha\beta} b'_{\beta\alpha}(\beta)\right) \mu(U'_{\alpha\beta})\mu(b'_{\alpha\beta}) \quad . \tag{4.85}$$

Now $\mu(b'_{\alpha\beta})$ is equal to $\det\left(O(\alpha)\right)\mu(b_{\alpha\beta})$ and so is equal to $\mu(b_{\alpha\beta})$. The Haar measure of $SO(n)$ is right and left invariant so $\mu(U'_{\alpha\beta}) = \mu(U_{\alpha\beta})$. By the equations (4.26), we have that the two delta of (4.85) can be written in the following form

$$\begin{gathered} \delta\left(O(\alpha)\big(\sum_{\beta=1}^{n+1} b_{\alpha\beta}(\alpha)\big)\right) \\ \delta\left(O(\alpha)\big(b_{\alpha\beta}(\alpha) - U_{\alpha\beta} b_{\beta\alpha}(\beta)\big)\right) \end{gathered} \tag{4.86}$$

that, from the properties of the delta function, will reduce to 1 over the determinant of $O(\alpha)$ for the deltas of the quantities in parenthesis that multiply $O(\alpha)$. Then we have that equation (4.85) can be written as

$$\delta\left(\sum_{\beta=1}^{n+1} b_{\alpha\beta}(\alpha)\right)\delta\left(b_{\alpha\beta}(\alpha)-U_{\alpha\beta}b_{\beta\alpha}(\beta)\right)\mu(U_{\alpha\beta})\mu(b_{\alpha\beta}) \quad , \tag{4.87}$$

and then the invariance of the measure under gauge transformations.

4.8 Coupling with Matter

In the continuum theory on Riemannian manifolds the coupling with fermionic matter is given by the following term of the Lagrangian density

$$\mathcal{L} \equiv \left(\bar{\psi}e^{\mu}_{a}\gamma^{a}\nabla_{\mu}\psi - \nabla_{\mu}e^{\mu}_{a}\bar{\psi}\gamma^{a}\psi\right) + m\psi\bar{\psi} \tag{4.88}$$

where γ^a, $a = 1, ...n$ are the Dirac-matrices satisfying the Clifford algebra

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\delta^{a\,b} \tag{4.89}$$

ψ the n-dimensional Dirac spinor field ($\bar{\psi} \equiv \psi^{\dagger}\gamma^1$),$\nabla_\mu$ the covariant derivative, e^{μ}_{a} the n-beins that is, on the tangent space of the Riemannian manifolds (M, g) where the Lagrangian density (4.88) is defined, the vector fields such that

$$g^{\mu\nu}(x) = e^{\mu}_{a}(x)e^{\nu}_{b}(x)\delta^{ab} \quad . \tag{4.90}$$

We are assuming that the Riemannian manifolds (M, g) in question have a spin structure, that is the second Stiefel-Whitney class is zero.

We have now all the ingredients to define the coupling of gravity with fermionic matter on the lattice in analogy with continuum case (for related works see [8] [88] [89] [90]).
Let $\nu = 2n$ or $\nu = 2n + 1$ (depending if n is even or odd) and consider the 2^ν dimensional two-fold covering group of $SO(n)$. So instead to consider the connection matrices $U_{\alpha\beta}$, we have to consider the $2^\nu \times 2^\nu$ connection matrices $D_{\alpha\beta}$ such that

$$D_{\alpha\beta}\gamma^a D^{-1}_{\alpha\beta} = (U_{\alpha\beta})^a_b\gamma^b \tag{4.91}$$

where in the left hand part of (4.91) we have used matrix notation. Given $D_{\alpha\beta}$ we determine $U_{\alpha\beta}$, but if we know $U_{\alpha\beta}$ we determine $D_{\alpha\beta}$ up to a sign. In particular, as it is easy to derive from (4.91), we can write $U_{\alpha\beta}$ as ([8])

$$(U_{\alpha\beta})_{ab} = \frac{1}{2^{\nu}} \mathrm{Tr}(\gamma_a D_{\alpha\beta} \gamma_b) \tag{4.92}$$

In the discrete theory we assume that the spinor field is a 2^{ν} complex vector defined in each vertices of the dual simplicial complex, that is to say a map that to each vertex α associates the 2^{ν} complex vector $\psi(\alpha)$.
In order to define the covariant derivative on lattice we have to derive the distance $|\alpha\beta|$ between the two neighborly circumcenters in α and β. The distance $\triangle h_1$ of the circumcenter in α from the face $\alpha\beta$ can be calculated by deriving the volume of the n-dimensional simplex obtained by joining the circumcenter with the n-vertices of the face $\alpha\beta$ and dividing it by n and the volume of the face itself. So we have that

$$\triangle h_1 = \frac{1}{n^2} \frac{\sum_{i=1}^{n} b^a_{\alpha\beta}(\alpha) z^i_a(\alpha)}{|b_{\alpha\beta}(\alpha)|} \tag{4.93}$$

in which z_i^a are the circumcentric coordinates of the vertices of the face $\alpha\beta$ and with $|b_{\alpha\beta}(\alpha)|$ the module of the vector that divided by $(n-1)!$ is equal to the volume of the face.
In the same manner we have

$$\triangle h_2 = \frac{1}{n^2} \frac{\sum_{i=1}^{n} b^a_{\beta\alpha}(\beta) z^i_a(\beta)}{|b_{\beta\alpha}(\beta)|} \tag{4.94}$$

At the end we have $|\alpha\beta| = \triangle h_1 + \triangle h_2$.

At this point we are ready to define the covariant derivative $(\nabla_{\mu}\psi)(\alpha)$ on lattice

$$(\nabla_{\mu}\psi)(\alpha) \equiv \frac{D_{\alpha\beta}\psi(\beta) - \psi(\alpha)}{|\alpha\beta|} \quad . \tag{4.95}$$

So far the discrete version of the the action for the coupling between gravity and fermionic matter can be written in the form

$$\begin{aligned} S_F \equiv & \sum_{\alpha} \Big(\sum_{(\alpha\beta),\beta=1,\dots,n+1} \frac{1}{|\alpha\beta|} (\bar{\psi}(\alpha) b^a_{\alpha\beta} \gamma_a D_{\alpha\beta} \psi(\beta) \\ & - D_{\alpha\beta} \bar{\psi}(\beta) b^a_{\alpha\beta} \gamma_a \psi(\alpha)) + m \bar{\psi}(\alpha) \psi(\alpha) \Big) \end{aligned} \tag{4.96}$$

so that the quantum measure that includes also fermionic matter can be written as

$$\begin{aligned} Z \;=\; & \int e^{-(S+S_F)} \prod_{\alpha} \mu\Big(\psi(\alpha)\Big)\mu\Big(\bar{\psi}(\alpha)\Big)\delta\left(\sum_{\beta=1}^{n+1} b_{\alpha\beta}(\alpha)\right) \\ & \prod_{\alpha\beta} \delta\left(b_{\alpha\beta}(\alpha) - U_{\alpha\beta} b_{\beta\alpha}(\beta)\right) \mu(D_{\alpha\beta})\mu(b_{\alpha\beta}) \end{aligned} \tag{4.97}$$

where S is the action for pure gravity (4.24), and $\mu(D_{\alpha\beta})$ the Haar measure on the two-fold covering group of $SO(n)$ while $\mu(\psi(\alpha)) = d\psi(\alpha)$ the standard measure on C^{2^ν}.

The coupling with Fermionic matter on lattice presented in this paragraph is quite similar to the one introduced in reference [89]. The differences consist of the fact that the action is here group theoretical and not Regge, and this formalism implies that from the beginnings the coupling is written on the dual Voronoï complex.

Appendix A

Branched Polymers

The notion of Branched Polymers is crucial in the classification of the so called "Elongated Phase" of 4-D simplicial quantum gravity. We are now going to review [56] some basic concepts of statistical mechanics on graphs.

An abstract *graph* is a set of points called "vertices" connected by lines. These lines introduce a natural concept of connectedness. A graph can be seen as a one complex. From now on all our graphs will be abstract, not embedded in some affine space so that we may omit the adjective abstract. A graph is *simple* if two distinct vertices are connected at most by only one line and the vertices at the end of each line are always distinct. A graph that is connected and simple is called a *tree*. The number of lines $l(i)$ at the vertex i is called "the coordination number" of the vertex i. In a tree with n vertices we have that $\sum_{i=1}^{n} l(i) = 2(n-1)$. A tree is *rooted* if we label one vertex which is called the *root*.

Two trees T and T' are combinatorial equivalent if there is a one to one map ϕ between them such that the vertices of T are mapped uniquely into the vertices of T' so that a line joining two vertices of T', ν'_i and ν'_j, is a line of T' if and only if it is the image of a line of T by ϕ, in the sense that there exist two vertices of T ν_i and ν_j joined by a line of T whose images by ϕ are ν'_i and ν'_j and the line joining them is mapped into the line joining ν'_i and ν'_j. If the two trees are rooted they are combinatorially equivalent if there is a one to one map such that together with the properties described before it maps the root of T into the root of T' and viceversa. In statistical mechanics tree graphs are called *Branched Polymers*. The statistical mechanics of these abstract tree graphs is also known as *mean field theory* of Branched Polymers to distinguish it from the case of embedded Branched Polymers.

Let's define as $r^{\alpha}(n)$ the number of combinatorial inequivalent rooted tree

graphs with n vertices in which the root has coordination number equal to one and all others vertices have coordination number less than or equal α. By Caley's formula we have

$$r^{\alpha}(n) = \sum_{\substack{l_1 \ldots l_n = 1 \\ \sum_{i=2}^{n} l_i = 2(n-1)-1}}^{\alpha} \frac{1}{(n-1)} \prod_{i=2}^{n} \frac{1}{(l_i - 1)!} \underset{\alpha \mapsto \infty}{\longrightarrow} \frac{(n-1)^{n-2}}{(n-1)!} \quad . \tag{A.1}$$

Let $\xi^{\alpha}(n)$ indicates the number of combinatorial inequivalent tree graphs with n vertices whose coordination number is less than or equal to α. We have that

$$\xi^{\alpha}(n) = \sum_{\substack{l_1 \ldots l_n = 1 \\ \sum_{i=1}^{n} l_i = 2(n-1)}}^{\alpha} \frac{1}{n(n-1)} \prod_{i=1}^{n} \frac{1}{(l_i - 1)!} \underset{\alpha \mapsto \infty}{\longrightarrow} \frac{n^{n-2}}{n!} \quad . \tag{A.2}$$

From the above formula or from geometrical arguments it is easy to see that

$$\xi^{\alpha}(n) = \frac{1}{n} r^{\alpha}(n+1) \quad . \tag{A.3}$$

The partition function $R^{\alpha}(\beta)$ over the ensemble of rooted tree graphs, whose vertices have coordination number less than or equal α, is defined in the following way

$$R^{\alpha}(\beta) \equiv \sum_{n=2}^{\infty} \beta^{n-1} r^{\alpha}(n) \quad , \tag{A.4}$$

and the analogous $Z^{\alpha}(\beta)$ unrooted

$$Z^{\alpha}(\beta) \equiv \sum_{n=1}^{\infty} \beta^{n} r^{\alpha}(n) \quad . \tag{A.5}$$

From these definitions and the identity A.3 we have that

$$\frac{d}{d\beta} Z^{\alpha}(\beta) = \frac{1}{\beta} R^{\alpha}(\beta) \quad . \tag{A.6}$$

As regard the definition of the Green function for tree graphs with coordination numbers up to α, the strategy is completely similar to the same case for dynamical triangulations (c.f. section 3.51). A path between two vertices of a tree graph is given by a

sequence $\{i\}_{i=1}^{i=l}$ of vertices such that the vertices j and $j+1$ ($j+1 \neq j$ and $j = 1, ..., l-1$) are joined by a line and i_1 coincides with the first vertex and i_l

with the second vertex. The length of the path is the number of its line, $l-1$ in this specific example. The distance between two vertices of a tree graph is the length of the minimal path that joins them.
As usual the micro-canonical Green function is the sum over all combinatorial inequivalent tree graphs $T^{\alpha}(r,n)$ whose vertices have coordination numbers up to α with n vertices and with two labelled vertex at distance r

$$G^{\alpha}(r,n) \equiv \sum_{T^{\alpha}(r,n)} \tag{A.7}$$

and the relative grand-canonical partition function is

$$G^{\alpha}(r,\beta) \equiv \sum_{n=1}^{\infty} \beta^{n} G^{\alpha}(r,n) \quad . \tag{A.8}$$

The relative susceptibility function is defined as follows

$$\chi^{\alpha}(\beta) \equiv \sum_{r=0}^{\infty} G^{\alpha}(r,\beta) \tag{A.9}$$

As for the dynamical triangulations in the definition above we can exchange the sum over r with the sum over n so that

$$\sum_{r} \sum_{T^{\alpha}(r,n)} \tag{A.10}$$

is the sum of all combinatorial inequivalent tree graphs with coordination number up to α and with two labelled vertices. Since for large n asymmetric tree graphs prevails this sum is asymptotical equal to $n^2 G^{\alpha}(r,n)$, so that as in 2.29 the susceptibility function is asymptotically equal to

$$\chi^{\alpha}(\beta) \asymp \frac{d^2}{d\beta^2} Z^{\alpha}(\beta) \quad . \tag{A.11}$$

Now we are going to use the following recursion relation

Lemma A.0.1 *The function* $r^{\alpha}(n)$ *can be written as*

$$r^{\alpha}(n) = \sum_{\gamma=0}^{\alpha-1} \frac{1}{\gamma!} \sum_{\substack{n_1,\ldots,n_{\gamma}\geq 2 \\ \sum_{i=1}^{\gamma} n_i = n+\gamma-2}} \prod_{i=1}^{\gamma} r^{\alpha}(n_i) \quad . \tag{A.12}$$

This recursion formula can be proved straightforwardly or by using formula A.1 or by geometric considerations. Anyway from this lemma we have the following evident corollary

Corollary A.0.1

$$R^{\alpha}(\beta) = \beta \sum_{\gamma=0}^{\alpha-1} \frac{1}{\gamma!}[R^{\alpha}(\beta)]^{\gamma} \tag{A.13}$$

At this point A.13 can be used to derive the differential equation for $R^{\alpha}(\beta)$,

$$\frac{d}{d\beta}R^{\alpha}(\beta) = \frac{R^{\alpha}(\beta)}{\beta}\left[1 + \beta\frac{(R^{\alpha}(\beta))^{\alpha-1}}{(\alpha-1)!} - R^{\alpha}(\beta)\right]^{-1} \tag{A.14}$$

This equation is divergent at the point β^c where the denominator vanish, that is to say

$$R^{\alpha}(\beta^c) = 1 + \beta^c\frac{(R^{\alpha}(\beta))^{\alpha-1}}{(\alpha-1)!} \tag{A.15}$$

This equation has only one solution for each value of α. Since $\frac{d}{d\beta}R^{\alpha}(\beta)$ is divergent in β^c we have that

$$\left.\frac{d\beta(R^{\alpha})}{dR^{\alpha}}\right|_{R^{\alpha}(\beta^c)} = 0 \quad . \tag{A.16}$$

By the implicit function theorem $\beta(R^{\alpha})$ result to be analytic around $R^{\alpha}(\beta^c)$ so that we can expand it in Taylor series

$$\beta(R^{\alpha}) - \beta^c = \frac{1}{2}\frac{d^2\beta(R^{\alpha})}{dR^{\alpha 2}}(R^{\alpha} - R^{\alpha}(\beta^c))^2 + o\left((R^{\alpha} - R^{\alpha}(\beta^c))^2\right) \quad , \tag{A.17}$$

and for β near β^c we get

$$R^{\alpha}(\beta) \asymp R^{\alpha}(\beta^c) + C(\beta^c - \beta)^{\frac{1}{2}} \tag{A.18}$$

These considerations together with equations A.6 and A.11 imply that for β near β^c

$$\chi^{\alpha}(\beta) \asymp (\beta^c - \beta)^{-\frac{1}{2}} \tag{A.19}$$

so that the susceptibility of Branched Polymers has the same exponent $\gamma_s = \frac{1}{2}$ of the mean field.

Appendix B

Baby Universes

The concept of Baby Universe in dynamical triangulations has been introduced for the first time by S. Jain and S.D.Mathur [57]. The original idea of these authors has been to introduce a way for describing the roughness of triangulated surfaces. Successively the average number of Baby Universes has been used as a sort of order parameter to classify the two phases, *crumpled* and *branched polymer* phases which have been seen for the first time in the Monte Carlo simulations of 4-D dynamical triangulations. In fact in the articles [48] and [52] numerical simulations have shown very few baby universes in the crumpled phase and a cascade of baby universes in the branched polymer phase. Motivated by these data it has been tried to put in correspondence the baby universes with the branched polymers. The deep reason of this correlation is discussed in chapter 3. Anyway seminal arguments in favor of the connection between branched polymers and baby universes of minimal size can be found in the article [58].

Let's give the main definitions regarding Baby Universes in the spirit of the article [57]. We shall restrict, for convenience, to the four dimensions and to the case in which the baby universe has the neck of minimum size (Minbu). A Minbu can be considered as a triangulation of a PL manifold in which we can distinguish two parts: the Mother universe that is the part that has the majority of the volume of the triangulation, and the baby universe which is the part with the minority of the volume. The two parts are glued together by the boundary of a four-simplex that is called the *neck* of the baby universe.

Now we are going to introduce, following [57], a way to enumerate the average number of baby universes with given volume V in the canonical ensemble of all triangulations of a PL-manifold with N_4 four-dimensional simplices. Anyway we want to stress that this enumeration is not rigorous but it is in good agreement

with the numerical results [48], [52], [55]. Suppose that the topology is that one of the sphere S^4 since our future discussions will be concentrated on this case. Imagine to separate the mother universe and the baby. We will obtain two triangulations of the sphere S^4 with a boundary made by the boundary of a four-simplex. The mother has $N_4 - V$ four dimensional simplices and N_2' two-dimensional simplices, the baby V four-dimensional simplices and N_2'' two-dimensional simplices. Obviously the sum $N_2' + N_2''$ will be equal to $N_2 + 10$, where N_2 is the number of two-dimensional simplices of the original triangulation, because in separating the mother and the baby we are counting the bone of the neck twice. Suppose now to fill the hole of the boundaries by gluing a four simplex to each of them. So that the number of combinatorial inequivalent triangulations of the mother universe is now $\rho_a(S^4, N_4 - V + 1, N_2')$ (c.f. 2.8) and that one of the baby is $\rho_a(S^4, V+1, N_2'')$. The *antsaz* is that all the combinatorial inequivalent configuration with a Minbu are obtained by marking a four simplex in the mother universe and one in the baby, and by gluing them through identifying the two marked simplices. We suppose, as usual, that we are in the regime of large N_4 so that asymmetric triangulations prevail. This implies that the marking process will produce $(N_4 - V + 1)\rho_a(S^4, N_4 - V + 1, N_2')$ combinatorial inequivalent triangulations for the mother universe and $(V+1)\rho_a(S^4, V+1, N_2'')$ for the baby universes. So that, at the end, the number of the combinatorial inequivalent Minbu $M(S^4, N_4, N_2, V)$ is

$$\begin{aligned} M(S^4, N_4, N_2, V) = \quad & 5!(N_4 - V + 1)(V+1) \\ & \times \rho_a(S^4, N_4 - V + 1, N_2')\rho_a(S^4, V+1, N_2'') \quad . \end{aligned} \tag{B.1}$$

where 5! comes from the fact that in the case of asymmetrical triangulations there are 5! combinatorial inequivalent ways of joining two four-simplices. Any Minbu of the characteristics defined above can be obtained in this way, so that B.2 really can be considered as a formula that gives an estimate of the number of all Minbus of baby-size V and number of bone N_2'' in the micro-canonical ensemble of all triangulations of S^4 with N_4 four-simplices and N_2 bones. Consider a triangulation T of S^4 with N_4 four-simplices and N_2 bones, which has a Minbu of size V and N_2'' bones. Let us indicate by $n_T(S^4, N_4, N_2, V)$ the number of combinatorial inequivalent triangulations that can be obtained from a given triangulation T by cutting a baby universe and attach it on another four simplex of T, previously marked, as described above. It is clear that $M(S^4, N_4, N_2, V) = \sum_T n_T(S^4, N_4, N_2, V)$, so that the average number of combinatorial inequivalent Minbu in the micro-canonical en-

samble can be approximated by dividing $M(S^4, N_4, N_2, V)$ by the micro-canonical partition function $\rho_a(S^4, N_4, N_2)$

$$\mathcal{N}(S^4, N_4, N_2, V) \approx 5!(N_4 - V + 1)(V + 1) \times \frac{\rho_a(S^4, N_4 - V + 1, N_2')\rho_a(S^4, V + 1, N_2'')}{\rho_a(S^4, N_4, N_2)} \quad . \tag{B.2}$$

In the canonical ensemble the average number of Minbu with volume V can be calculated in the following way

$$\mathcal{N}(S^4, N_4, k_2, V) \approx 5!(N_4 - V + 1)(V + 1) \times \frac{\sum_{N_2} e^{k_2 N_2} \rho_a(S^4, N_4 - V + 1, N_2')\rho_a(S^4, V + 1, N_2'')}{Z(S^4, N_2, k_2)} \quad , \tag{B.3}$$

which using the relation $N_2' + N_2'' = N_2 + 10$ is equal to

$$\mathcal{N}(S^4, N_4, k_2, V) \approx 5!(N_4 - V + 1)(V + 1)e^{-10k_2} \times \frac{\sum_{N_2'} e^{k_2 N_2'} \rho_a(S^4, N_4 - V + 1, N_2') \sum_{N_2''} e^{k_2 N_2''} \rho_a(S^4, V + 1, N_2'')}{Z(S^4, N_2, k_2)} \quad . \tag{B.4}$$

Finally

$$\mathcal{N}(S^4, N_4, k_2, V) \approx 5!(N_4 - V + 1)(V + 1)e^{-10k_2} \times \frac{Z(S^4, N_4 - V + 1, k_2)Z(S^4, V + 1, k_2)}{Z(S^4, N_2, k_2)} \quad . \tag{B.5}$$

For large value of N_4 and V the previous formula will reduce to the standard formula for the average number of baby universe in the canonical ensemble

$$\mathcal{N}(S^4, N_4, k_2, V) \approx \frac{5!(N_4 - V + 1)(V + 1)Z(S^4, N_4 - V, k_2)Z(S^4, V, k_2)}{Z(S^4, N_2, k_2)} \quad . \tag{B.6}$$

We remember that in each phase the canonical partition function of A four-simplices has the behavior $Z(S^4, A, k_2) \asymp A^{\gamma_{str} - 3} e^{A k_4(k_2)}$, which substituted in B.6 gives

$$\mathcal{N}(S^4, N_4, k_2, V) \approx N_4 \left(V\left(1 - \frac{V}{N_4}\right) \right)^{\gamma_{str}-2} \quad . \tag{B.7}$$

In this way by computing the average number of baby universes with volume V in a canonical ensamble of N_4 simplices we can obtain informations on the susceptibility exponent γ_{str}. This is the reason too for which the argument of the baby universes is so popular among people doing numerical simulations.

Appendix C

Voronoï Cells

In this section we shall introduce the definition of Voronoï Cells since we are going to use it for the definition of the "metric" dual of a triangulation of a PL-manifold.

Consider a point set Ξ of a n-dimensional Euclidean space E^n. We say that E^n is discrete if there is a positive real number r_0 such that $\forall x, y \in E^n$ $d(x, y) \geq 0$, where d is the distance function between two points of E^n induced by the Euclidean metric [70].
It follows immediately from the above definition this theorem:

Theorem C.0.1 *If Ξ is discrete it has the local finiteness property: for every closed ball $\overline{B}_r(x)$ with center in $x \in E^n$ and radius r, $\overline{B}_r(x) \cap \Xi$ is a finite set.*

A point set Ξ is relative dense in E^n if there is a real positive number R_0 such that all the spheres of E^n of radius greater than R_0 have at least one point of Ξ in their interior. R_0 is called the covering radius of Ξ.

A set point Ξ which is discrete and relative dense is a Delaunay or Delone set [70] [71], and is also indicated as (r, R) from its discrete radius r and covering radius R.
In a Delone set Ξ the r-star, $r > 0$, of a point $x \in \Xi$ is the finite point set $\overline{B}_r(x) \cap \Xi$.

Consider now a Delone set $\Xi \in E^n$ (or, equivalently, a set with a finite number of points). Let $x \in \Xi$, the Voronoï cell of x is, by definition, the convex region of points of E^n which are more close to x than to any other point of Ξ.
In practice a Voronoï cell of a point $x \in \Xi$ is constructed by considering the segments that join x to each point of Ξ in the star of x. Anyway for the moment we are considering the case in which Ξ is a sort of periodic crystal so that the radius of the star has to be chosen in such a way that it contains x and the points of its elementary cell (see figure C.1).

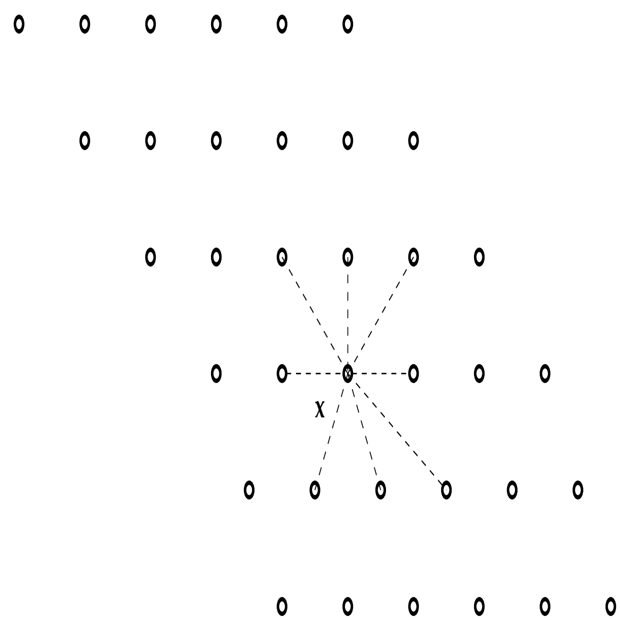

Figure C.1: In this regular crystal the points to be considered for constructing the Voronoï cell $V(x)$ of x are those ones which belong to the star of x made by the points closer to x. These points are also the points which form an elementary cell of this crystal

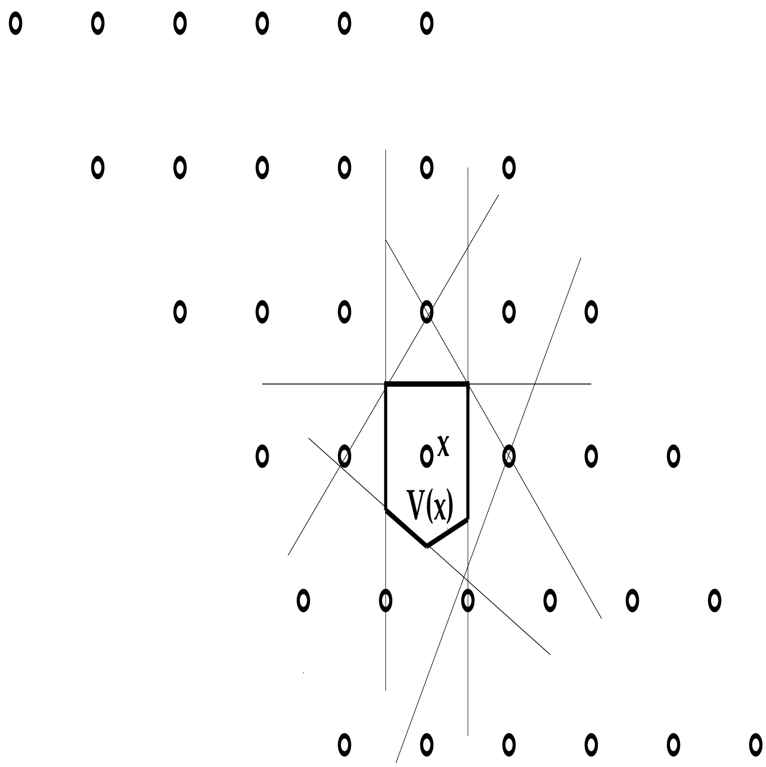

Figure C.2: Voronoï cell $V(x)$ of the point x belonging to the crystal of the figure C.1

Then the $n-1$-dimensional faces of the Voronoï cell of x $V(x)$, called facets, are obtained by considering the $n-1$-dimensional orthogonal bisectors to the segments. The intersection of the facets $V(x)$ are the $n-2$ dimensional face of the Voronoï cell and so on (see figure C.2).
If we carry on this construction for each point of Ξ we obtain a partition of E^n into cells that is called *Voronoï tessellation*. This tessellation has the following properties:
1)The Voronoï cells are convex region of E^n and two cells can have in common only one $n-1$ dimensional facets.
2)The points of Ξ whose Voronoï cells share a vertex v lie on a sphere with center in v which has no points of Ξ in its interior. Furthermore it is easy to see that the sphere centered in v pass among at least $n+1$ points of Ξ.

If Ξ has an infinite number of points and it doesn't have any periodic structure, for constructing a Voronoï cell of a point $x \in \Xi$ in the general case we need an infinite number of operations. But we are going to show that really they are still

finite, thanks to the following theorem

Theorem C.0.2 *Let Ξ a Delone set in E^n of type (r, R) in the previous notations. To construct a Voronoï cell in E^n of the point $x \in \Xi$ we have to join x to all point of Ξ in the ball $\overline{B}_{2R}(x)$.*

Dim:

Consider the Voronoï tessellation of E^n based on the Delone set Ξ. Let x a point of Ξ and $V(x)$ the facets of the Voronoï cell relative to x. By definition the $V(x)$ lie on the $n-1$ iperplains that are bisectors of the segment joining x to its neighbor points of Ξ. Let v any vertex of $V(x)$. We know that v is the center of a sphere on which there are at least n point of Ξ whose relative Voronoï cells intersect at v. In the interior of this sphere there are no points of Ξ so that $d(x, v) \leq R$. So that if we consider the distance between x and any other point y among all other at least n points above, we have $d(x, y) \leq d(x, v) + d(y, v)$ since the reasonings for x can be applied equivalently to any other point y, we have $d(y, v) \leq R$ and then $d(x, y) \leq 2R$. This implies that to construct a Voronoï cell in E^n of a point x belonging to a Delone

set it is sufficient to consider the points of Ξ which are in the ball $\overline{B}_{2R}(x)$.

The triangulations Σ of a PL-manifold, which we are considering, have always a finite number of n simplices so that they have a finite number of vertices. On Σ it is defined a metric tensor such that in the interior of each n-dimensional simplex it coincides with the Euclidean metric and near the hinges it is like the metric tensor of a cone. So it does make sense to define on the simplicial manifold the Voronoï cell

(see reference [20] p. 395) of a vertex as the convex region of points of the simplicial complex that are closer to the vertex than to any other vertex of the simplicial complex. To construct this Voronoï cell relative to a vertex we consider on the simplicial complex all the vertices that lie in the star of this vertex and the edges joining them with the vertex. The facets of this Voronoï cell lie on the $n-1$ iperplain orthogonal to the edges in their middle points. Standard facts of elementary Euclidean geometry tell us that the vertices of the Voronoï cells coincides with the circumcenters of the simplices, that is the centers of the spheres circumscribed to the simplices. In this way we have constructed the dual of the original triangulation Σ. In fact to each vertex uniquely it corresponds a n-dimensional Voronoï cell, to an edge a $n-1$ dimensional facet of the Voronoï cell and in general to a k simplex a $n-k$-Voronoï polyhedron orthogonal to it. In particular this dual application maps a n-simplex to its circumcenter.

Bibliography

[1] Tullio Regge, General Relativity without Coordinates, Nuovo Cimento, vol. 19, number 3, pag. 558–571, 1961.

[2] G.W. Gibbons and S.W. Hawking, Euclidean Quantum Gravity, World Scientific, 1993.

[3] Giampiero Esposito, Alexander Yu. Kamenshchik, Giuseppe Pollifrone, Euclidean Quantum Gravity on Manifolds with Boundary, Kluwer-Fundamental Theories of Physics, vol. 85, 1997.

[4] Stephen Hawking, Spacetime Foam, Nuclear Physics B, vol. 144, pag. 349–362, 1978.

[5] William P.Thurston, Three Dimensional Geometry and Topology, M.S.R.I., 1991.

[6] H. Seifert and W. Threlfall, A Textbook of topology, Academic Press, 1980.

[7] C.P. Rourke and B.J. Sanderson, Introduction to Piecewise-Linear Topology, Springer-Verlag, 1972.

[8] Jürg Fröhlich, Non-Perturbative Quantum Field Theory, Regge Calculus and Discretized Gravitational Functional Integrals, World Scientific, Singapore, 1992, pag. 523–545.

[9] , William P. Thurston, Shapes of Polyhedra and Triangulations of the Sphere, math.GT/9801088, 1998.

[10] Bryce DeWitt, Relativity, Groups and Topology II, Les Houches 1983 Session XL, The Spacetime Approach To Quantum Field Theory, Bryce Dewitt and Raymond Stora, North-Holland, pag. 381–738, 1984.

[11] Herbert W. Hamber, Critical Phenomena, Random Systems, Gauge Theories, Les Houches 1984 Session XLII, Simplicial Quantum Gravity, K. Osterwalder and R. Stora, North-Holland, pag. 375–439, 1986.

[12] M. Rocek and R. Williams, Quantum Regge Calculus, Physics Letter B, vol. 104, pag. 31–37, 1981.

[13] M. Rocek and R. Williams, The Quantizzation of Regge Calculus, Z. Phys. C, vol. 21, 371–381, 1984.

[14] Herbert W. Hamber and Ruth Williams, Gauge Invariance in Simplicial Gravity , Nuclear Physics B, vol. 487, pag. 345–408, 1997.

[15] Herbert W. Hamber, Phases of simplicial quantum gravity in four dimensions: estimates for the critical exponents , Nuclear Physics B, vol. 400, 347–389, 1993.

[16] Herbert W. Hamber, Critical behavior in simplicial quantum gravity , Nuclear Physics Proc. Suppl., vol.20, pag.728–732, 1991.

[17] Herbert W. Hamber, Phases of 4D Simplicial Quantum Gravity, Physical Review D, vol. 45, pag. 507–512, 1992.

[18] , R. Friedberg and T.D. Lee, Derivation of Regge's Action from Einstein's Theory of General Relativity, Nuclear Physics B, vol. 242, 145–166, 1984.

[19] G. Friedberg, R. Friedberg, T.D. Lee, H.C. Ren, Lattice Gravity Near the Continuum Limit, Nuclear Physics B, vol. 245, 343–368, 1984.

[20] Herbert W. Hamber and Ruth Williams, Higher Derivative Quantum Gravity on a Simplicial Lattice , Nuclear Physics B, vol. 269, 712–734, 1984.

[21] Renate Loll, Discrete Approaches to Quantum Gravity in Four Dimensions, Living Review, http://www.livingreview.org/.–gr-qc/9805049, 1998.

[22] B. Brügmann and E. Marinari, Monte Carlo simulations of 4d Simplicial Quantum Gravity, Journal of Mathematical Physics, vol. 36, pag. 6340–6352, 1995.

[23] Ruth Williams and Philip A. Tuckey, Regge Calculus: a Bibliography and a Brief Review, Classical and Quantum Gravity, vol. 9, pag. 1409–1422, 1992.

[24] James B. Hartle, Warner Miller and Ruth Williams , Signature of the simplicial supermetric, Classical and Quantum Gravity, vol. 14, pag. 2137–2155, 1997.

[25] Weingarten, D, Euclidean Quantum Gravity on a Lattice, Nuclear Physics B, vol. 210, pag. 229–245, 1982.

[26] Herbert W. Hamber and Ruth M. Williams, On the Measure in Simplicial Gravity, hep-th/9708019, 1997.

[27] A. Jevicki and M. Ninomiya, Functional Formulation of Regge Gravity , Physical Review D, vol. 33, pag. 1634–1637, 1986.

[28] Claude Itzykson and Jean-Michel Drouffe, Statistical Field Theory, vol. 1, Cambridge University Press, 1989.

[29] H. Römer and M. Zähringer, Functional Integration and Diffeomorphism Group in Euclidean Lattice Quantum Gravity, Classical and Quantum Gravity, vol. 3, 897–910, 1986.

[30] Jan Ambjørn, Mauro Carfora, Annalisa Marzuoli, The Geometry od Dynamical Triangulations, Lecture Notes in Physics m50, Springer-Verlag, 1997.

[31] C. Bartocci, U. Bruzzo, M. Carfora, A. Marzuoli, Entropy of Random Coverings and 4D Quantum Gravity, Journal of Geometry and Physics, vol. 18, pag. 247–294, 1996.

[32] M. Carfora, A. Marzuoli, Holonomy and Entropy Estimates for Dynamical Triangulated Manifolds, Journal of Mathematical Physics, vol. 36, 6353–6376, 1995.

[33] M. Carfora, A. Marzuoli, Entropy Estimates for Simplicial Quantum Gravity, Journal of Geometry and Physics, vol. 16, 99–119, 1996.

[34] , K. Osterwalder and R. Schrader, Aximos for Euclidean Green's Functions, Comunications in Mathematical Physics, vol. 42, pag. 83–112, 1973.

[35] K. Osterwalder and R. Schrader, Aximos for Euclidean Green's Functions.II., Comunications in Mathematical Physics, vol. 42, pag. 281–305, 1975.

[36] Roberto Fernández, Jürg Fröhlich, Alan D. Sokal, Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory, Springer-Verlag, 1992.

[37] R.F. Streater and A. S. Wightman, PCT, Spin and Statistics, and All That, W.A. Benjamin, 1964.

[38] Franco Strocchi, Selected Topics on the General Properties of Quantum Field Theory, Lecture notes in Physics vol.51, World-Scientific, 1993.

[39] B.A. Dubrovin, A.T. Fomenko, S.P. Novikov, Modern Geometry-Methods and Applications vol.3, Springer-Verlag, 1984.

[40] W.T. Tutte, A Census of Planar Triangulations, Canadian Journal of Mathematics, vol. =14, pag. 21–38, 1962.

[41] D. Bessis, C. Itzykson, J.B. Zuber, Quantum Field Theory Techniques in Graphical Enumeration, Advances in Applied Mathematics, vol. 1, pag. 109–157, 1980.

[42] E. Brèzin, C. Itzykson, G. Parisi, J. B. Zuber, Planar Diagrams, Communication in Mathematical Physics, vol. 59, pag. 35–51, 1978.

[43] Edward Witten, Two-Dimensional Gravity and Intersection Theory on Moduli space, Surveys in Differential Geometry, vol. 1, pag. 243–310, 1991.

[44] Ricardo Estrada, Ram P. Kanwal, Asymptotic Analysis: A Distributional Approach, Brikbäuser, 1994.

[45] Edward A. Bender and Nicolas C. Wormald, Almost All Convex Polyhedra Are Asymmetric, Canadian Journal of Mathematics, vol. 37, pag. 854–871, 1985.

[46] W.T. Tutte, On the Enumeration of Convex Polyhedra, Journal of Combinatorial Theory, Series B, vol. 28, pag. 105–126, 1980.

[47] L. B. Richmond and N.C. Wormald, The Asymptotic Number of Convex Polyhedra, Transactions of the American Mathematical Society, vol. 273, 721–735, 1982.

[48] J. Ambjørn, S. Jain, J. Jurkiewicz, C.F. Kristjansen, Observing 4d Baby Universes in Quantum Gravity, Physics Letters B, vol. 305, pag. 208–213, 1993.

[49] M. E. Agishtein and A. A. Migdal Critical behavior of dynamical triangulated quantum gravity in four-dimensions, Nuclear Physics B, vol. 385, 395–412, 1992.

[50] M. E. Agishtein and A. A. Migdal, Simulations of Four-Dimensional Simplicial Quantum Gravity, Modern Physics Letters A, vol. 7, pag. 1039–1061, 1992.

[51] J. Ambjørn, J. Jurkiewicz, Four-Dimensional Simplicial Quantum Gravity, Physics Letters B, vol. 335, pag. 355–358, 1992.

[52] J. Ambjørn, J. Jurkiewicz, Scaling in Four-Dimensional Quantum Gravity, Nuclear Physics B, volume =451, 643–676, 1995.

[53] Bastiaan Valentijn de Bakker, Further Evidence that the Transition of 4-D Dynamical Triangulations is First Order, Phys. Lett. B, vol. 389, pag. 238–242, 1996.

[54] P. Bialas, Z. Burda, A. Kryzywicki, B. Petersson , Focusing on the Fixed Point of 4D Simplicial Gravity, Nucl. Phys. B, hep-lat/9601024, vol. 472, 293–308, 1996.

[55] S. Catteral, J. Kogut R. Renken, G. Thorleifsson, Baby Universes in 4D Dynamical Triangulations, Physics Letters B, vol. 366, pag. 72–76 1996.

[56] Jürg Fröhlich, Critical Phenomena, Random Systems, Gauge Theories, Les Houches 1984 Session XLIII, Mathematical Aspects of Disordered Systems, North-Holland, 1986, pag. 728–893.

[57] Sanjay Jain and Samir D. Mathur, World-Sheet Geometry and Baby Universes in 2D Quantum Gravity, Physics Letters B, vol. 286, pag. 239–246, 1992.

[58] Jan Ambjørn, Bergfinnur Durhuus and Thórdur Jónson , Summing over all Genera for d>1: a Toy Model, Physics Letters B, vol. 244, pag. 403–412, 1990.

[59] Bastiaan Valentijn de Bakker, 1995, Universiteit van Amsterdam Instituut voor Theoretische Fysica, Amsterdam Valckenierststraat 65 1018 XE, Simplicial Quantum Gravity, hep-lat/9508006 7 Aug 1995,

[60] David W. Walkup, The Lower Bound Conjecture for 3 and 4 Manifolds, Acta Mathematica, vol. 125, 75–107, 1970.

[61] J. Ambjørn, J. Jurkiewicz and Y. Watabiki , Dynamical Triangulations, a Gateway to Quantum Gravity?, Journal of Mathematical Physics, vol. 36, pag. 6299–6339, 1995.

[62] Richard Otter, The Number of Trees, Annals of Mathematics, vol. 49 pag. 583–599, 1948.

[63] B. Durhuus, J. Frölich, T. Jónsson, Critical Behaviour in a Model of Planar Random Surfaces, Nuclear physics B, vol. 240, 453–480, 1984.

[64] Jan Ambjørn, Fluctuating Geometries in Statistical Mechanics and Field Theory, Les Houches 1994 Session LXII, Quantizzation of Geometry, K. Osterwalder and R. Stora, North-Holland, pag. 77–193, 1996.

[65] Wolfang Künel, Advances in Differential Geometry and Topology, Triangulations of Manifolds with Few Vertices, F. Tricerri, World Scientific, Singapore, 59–115, 1990.

[66] F. David, Gravitation and Quantization, Les Houches 1992 Session LVII, Simplicial Quantum Gravity and Random Lattice, B. Julia and J. Zinn-Justin, North-Holland, 679-781, 1995.

[67] Jan Ambjørn, Bergfinnur Durhuus, Thórdur Jónson, Quantum Geometry, Cambridge University Press, 1997.

[68] Davide Gabrielli, Polymeric Phase of Simplicial Quantum Gravity, Physics Letters B, vol. 366, 79–85, 1998.

[69] Gabriele Gionti, The Geometry of the Elongated Phase in 4-D Simplicial Quantum Gravity, To appear in Journal of Mathematical Physics, hep-lat/9711018, 1998.

[70] Marjorie Senechal, Quasicrystal and Geometry, Cambridge University Press, 1995.

[71] A. Okabe, B. Boots, K. Sugihara, SPAZIAL TESSELLATIONS, Concepts and Applications of Voronoi Diagrams, John Wiley and Sons, 1992.

[72] Michele Caselle, Alessandro D'Adda, Lorenzo Magnea, Regge Calculus as A Local Theory of The Poincaré Group, Physics Letter B, vol. 232, pag. 457–461, 1989.

[73] Michele Caselle, Alessandro D'Adda, Lorenzo Magnea, Doubling of all matter fields coupled with gravity, Physics Letter B, vol. 192, pag. 411–414, 1987.

[74] , Michele Caselle, Alessandro D'Adda, Lorenzo Magnea, Lattice Gravity and Supergravity as a Spontaneously Broken Gauge Theories of the (Super) Poincaré Group, Physics Letter B, vol. 192, pag. 406–410, 1987.

[75] Alessandro D'Adda, Geometrical Approach to Gravity and Supergravity, Proceedings of Varenna 1980 *Theory of Fundamental Interactions, 268–299, 1982.

[76] Alessandro D'Adda, On The Construction of Gravity and Supergravity from Fundamental Spinorial Constituents, Physics Letter B, 119, 334–338, 1982.

[77] John Barrett, First Order Regge Calculus, Classical and Quantum Gravity, vol. 11, pag. 2723–2730, 1994.

[78] Warner A. Miller, The Hilbert Action in Regge Calculus, Classical and Quantum Gravity, vol. 14, pag. L199–L204, 1997.

[79] Noboru Kawamoto, Holger Bech Nielsen, Lattice Gauge Gravity, Physical Review D, vol. 43, pag. 1150–1156, 1991.

[80] Carlo Rovelli, Strings Loops and Others: a Critical Survey of the Present Approaches to Quantum Gravity, Plenary lecture on quantum gravity at GR15 conference, Pune, India–gr-qc/9803024, 1998.

[81] Rafael Sorkin, Forks in the Road, on the Way to Quantum Gravity, International Journal of Theoretical Physics–gr-qc/9706002, vol. 36, pag. 2759–2781, 1997.

[82] Rafael Sorkin, Time-Evolution problem in Regge Calculus, Physical Review D, vol. 12, pag. 385–396, 1975.

[83] Rafael Sorkin, The Electromagnetic Field on a Simplicial Net, Journal of Mathematical Physics, vol. 16, pag. 2432–2440, 1975.

[84] Kenneth G. Wilson, Confinament of Quarks, Physical Review D, vol. 10, pag. 2445–2459, 1974.

[85] A. Palatini, Deduzione Invariantiva delle Equazioni Gravitazionali dal Principio di Hamilton, Rendiconti del Circolo Matematico di Palermo, vol. 10, pag. 203, year 1919.

[86] Abhay Ashtekar , Lectures on Non-Perturbative Canonical Gravity, World Scientific, Singapore, 1991.

[87] Giampiero Esposito, Gabriele Gionti, Cosimo Stornaiolo, Space-Time Covariant Form of Ashtekar's Constraints, Nuovo Cimento B, vol. 110, pag. 1137–1151, 1995.

[88] I. T. Drummond, Regge-Palatini Calculus, Nuclear Physics B, vol. 273, pag. 125–136, 1986.

[89] Hai-cang REN, Matter Fields in Lattice Gravity, Nuclear Physics B, vol. 301, pag. 661–684, 1988.

[90] Alexander N. Jourjine, Dimensional Phase Transitions: Spinors, Gauge Fields, and Background Gravity on Cell Complex, Physical Review D, vol. 34, pag. 3058–3068, 1986.

[91] Alessandro D'Adda and Gabriele Gionti, Lattice Gauge Theory of Gravity in the First Order Formalism, To be submitted, 1998.

[92] Mauro Carfora, Davide Gabrielli, Gabriele Gionti, 12 Italian Conference on General Relativity and Gravitational Physics, Recent Developments in 4D-Simplicial Quantum Gravity, M. Bassan V. Ferrari M. Francaviglia F. Fucito I. Modena, World Scientific, Singapore, 111–126, 1997.